

Actions of compact groups, C^* -index theorem, and families

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Abstract

We prove the index theorem for elliptic operators acting on sections of bundles where fiber is equal to a projective module over a C^* -algebra, in the situation of action of a compact Lie group on this algebra as well as on the total space commuting with symbol. As an application the equivariant index theorem for a direct product of base by the space of parameters is obtained.

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1 Introduction

The rational version of index theorem for C^* -elliptic operators [29, 31] has numerous applications in differential topology (to proof the Novikov conjecture), differential geometry (curvature of spin manifolds), theory of elliptic operators with random and periodic coefficients, etc. After that the equivariant generalization of this theorem [41], as well as equivariant generalization taking the torsion into account [42] was obtained. They also have found their applications (see, e.g. [37, 39]). In these theorems a compact Lie group acts on the total space of a bundle of A -modules, but not on the algebra A itself. For some applications, the most classic of which is calculation of equivariant index of a family of elliptic operators, it is necessary for the group to act on A compatible with the action on the total space. The present paper is devoted to the decision of this problem.

We use intensively the foundations of theory of C^* -Hilbert modules. One can find it in [27, 45, 28]. The modern state of index theory for families and its applications can be found e.g. in [44, 10].

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2 An averaging theorem

Let G be a compact Lie group acting continuously on C^* -algebra A by involutive automorphisms. If A is unital, then the unity has to be invariant. Then $g \in G$ takes self-adjoint elements to self-adjoint and positive ones to positive.

Remark 2.1 If $a \geq b \geq 0$, then $ga \geq gb$. Indeed, $a - b = c^*c \geq 0$. Hence, $ga - gb = g(c)^*g(c) \geq 0$.

An A -module M is called *GGA-module* if it is equipped with a \mathbf{C} -linear action of G , such that

$$g(m \cdot a) = g(m) \cdot g(a), \quad m \in M, \quad a \in A.$$

Definition 2.2 An inner A -valued product on Hilbert module is called *invariant* or *GGA-product* if

$$\langle gx, gy \rangle = g(\langle x, y \rangle).$$

Let us remark that the averaging theorem of [41] (see also [39]) does not define an invariant product in the *GGA*-case. Moreover, the formula

$$\langle u, v \rangle = \int_G \langle T_x u, T_x v \rangle' dx,$$

does not define an A -inner product:

Example 2.3. Let $\mathcal{M} = A = C(G)$. Then the new “product” is valued in constants only. Some effects of such originality are studied in [15].

Definition 2.4 Let us define an action of G on the module $l_2(A)$ over a unital G -algebra A by the formula $g(u_1, u_2, \dots) = (gu_1, gu_2, \dots)$.

Lemma 2.5 *This is a continuous action, and the initial A -valued product is invariant with the respect to it.*

Proof: First of all

$$\begin{aligned} g(ua) &= g((u_1, u_2, \dots)a) = (g(u_1a), g(u_2a), \dots) = (g(u_1)g(a), g(u_2)g(a), \dots) = \\ &= (g(u_1), g(u_2), \dots)g(a) = g(u)g(a). \end{aligned}$$

Further,

$$\langle gu, gv \rangle = \sum_{i=1}^{\infty} (g(u_i))^* g(v_i) = \sum_{i=1}^{\infty} g(u_i^*) g(v_i) = \sum_{i=1}^{\infty} g(u_i^* v_i) = g\left(\sum_{i=1}^{\infty} u_i^* v_i\right) = g(\langle u, v \rangle).$$

Let us demonstrate the continuity. Let $\|g - h\| < \varepsilon$ as automorphisms of the algebra A . Then

$$\begin{aligned} \|\langle (g - h)u, (g - h)u \rangle\| &= \left\| \sum_{i=1}^{\infty} ((g - h)(u_i))^* (g - h)(u_i) \right\| = \left\| \sum_{i=1}^{\infty} (g - h)(u_i^* u_i) \right\| = \\ &= \left\| (g - h) \sum_{i=1}^{\infty} u_i^* u_i \right\| \leq \varepsilon \|u\|^2. \quad \square \end{aligned}$$

Let us recall some facts about the integration of operator-valued functions [25]. Let X be a compact space, A be a C^* -algebra, $\varphi : C(X) \rightarrow A$ be an involutive homomorphism of algebras with unity, and $F : X \rightarrow A$ be a continuous map, such that for every $x \in X$ the element $F(x)$ commutes with the image of φ . In this case the integral

$$\int_X F(x) d\varphi \in A$$

can be defined in the following way. Let $X = \cup_{i=1}^n U_i$ be an open covering and

$$\sum_{i=1}^n \alpha_i(x) = 1$$

be a corresponding partition of unity. Let us choose the points $\xi_i \in U_i$ and compose the integral sum

$$\sum(F, \{U_i\}, \{\alpha_i\}, \{\xi_i\}) = \sum_{i=1}^n F(\xi_i) \varphi(\alpha_i).$$

If there is a limit of such integral sums then it is called the corresponding integral.

If X is a Lie group G then it is natural to take φ equal to the Haar measure

$$\varphi : C(X) \rightarrow \mathbf{C}, \quad \varphi(\alpha) = \int_G \alpha(g) dg$$

(though this is only a positive linear map, not a $*$ -homomorphism) and to define for a norm-continuous $Q : G \rightarrow L(H)$ (A is realized as a subalgebra in algebra $L(H)$ of all bounded operators on Hilbert space H)

$$\int_G Q(g) dg = \lim \sum_i Q(\xi_i) \int_G \alpha_i(g) dg.$$

If $Q : G \rightarrow P^+(A) \subset L(H)$, then, since $\int_G \alpha_i(g) dg \geq 0$, we get

$$\sum_i Q(\xi_i) \cdot \int_G \alpha_i(g) dg \in P^+(A) \quad \text{and} \quad \int_G Q(g) dg \in P^+(A)$$

(the positive cone $P^+(A)$ is convex and closed). So we have proved the following Lemma.

Lemma 2.6 *Let $Q : G \rightarrow P^+(A)$ be a continuous function. Then for the integral in the sense of [25] we have*

$$\int_G Q(g) dg \geq 0. \quad \square$$

Theorem 2.7 *Let $GL = GL(A)$ be the full general linear group, i. e., the group of all bounded A -linear automorphisms of $l_2(A)$. Let $g \mapsto T_g$ ($g \in G, T_g \in GL$) be a representation of G such that the map*

$$G \times l_2(A) \rightarrow l_2(A), \quad (g, u) \mapsto T_g u$$

is continuous. Then on $l_2(A)$ there is a GGA-product equivalent to the original one.

Proof: Let \langle, \rangle' be the original product. We have a continuous map $G \rightarrow A$, $x \mapsto x^{-1}(\langle T_x u, T_x v \rangle')$, for every u and v from $l_2(A)$. We define the new product by

$$\langle u, v \rangle = \int_G x^{-1}(\langle T_x u, T_x v \rangle') dx,$$

where the integral can be defined in the sense of either of the two definitions from [25, p. 810] because the map is continuous with the respect to the norm of the C^* -algebra. This product is an A -Hermitian map $l_2(A) \times l_2(A) \rightarrow A$. Indeed,

$$\begin{aligned} \langle u \cdot a, v \cdot b \rangle &= \int_G x^{-1}(\langle T_x(u \cdot a), T_x(v \cdot b) \rangle') dx = \\ &= \int_G x^{-1}(\langle T_x(u) \cdot (xa), T_x(v) \cdot (xb) \rangle') dx = \\ &= \int_G x^{-1}[(xa)^*(\langle T_x(u), T_x(v) \rangle')(xb)] dx = \\ &= \int_G x^{-1}[x(a^*)(\langle T_x(u), T_x(v) \rangle')(xb)] dx = \\ &= \int_G (a^*)x^{-1}(\langle T_x(u), T_x(v) \rangle')b dx = a^* \langle u, v \rangle b. \end{aligned}$$

Since $x^{-1} : A \rightarrow A$ is an involutive mapping, it takes positive elements to positive ones and by Lemma 2.6

$$\langle u, u \rangle = \int_G x^{-1}(\langle T_x u, T_x u \rangle') dx, \geq 0.$$

For $T_x u = (a_1(x), a_2(x), \dots) \in l_2(A)$ the equality $\langle u, u \rangle = 0$ takes the form

$$\int_G \sum_{i=1}^{\infty} x^{-1}(a_i(x)a_i^*(x)) dx = 0.$$

Since $x^{-1}(a_i(x)a_i^*(x)) \geq 0$, we have $a_i(x) = 0$ almost everywhere. Hence, $a_i(x) = 0$ for all x by the continuity, and $T_x u = 0$. In particular, $u = 0$.

Let us demonstrate the continuity of this new product. Fix $u \in l_2(A)$. Then by Lemma 2.5, $x \mapsto T_x(u)$, $G \rightarrow l_2(A)$ is a continuous mapping of a compact space. Hence, the set $\{T_x(u) \mid x \in G\}$ is bounded. So, by the principle of uniform boundness [9, vol. 2, p. 309]

$$\lim_{v \rightarrow 0} T_x(v) = 0 \quad (1)$$

uniformly with respect to $x \in G$. If u is fixed then

$$\|T_x(u)\| \leq M_u = \text{const}$$

and by (1)

$$\begin{aligned} \|\langle u, v \rangle\| &= \left\| \int_G x^{-1} (\langle T_x u, T_x v \rangle') dx \right\| \leq \\ &\leq M_u \cdot \text{vol } G \cdot \sup_{x \in G} \|T_x(v)\| \rightarrow 0 \quad (v \rightarrow 0). \end{aligned}$$

This gives the continuity at 0 and hence on the whole space $l_2(A) \times l_2(A)$. Since each operator T_y is an automorphism, we get

$$\begin{aligned} \langle T_y u, T_y v \rangle &= \int_G x^{-1} (\langle T_x T_y u, T_x T_y v \rangle') dx = \\ &= \int_G y(xy)^{-1} (\langle T_{xy} u, T_{xy} v \rangle') dx = \\ &= y(\langle u, v \rangle), \end{aligned}$$

i. e. this product is invariant.

Now we will show the equivalence of the two norms and, in particular, the continuity of the representation. There is a number $N > 0$ such that $\|T_x\|' \leq N$ for every $x \in G$. So by [25] (for the simplicity $\text{vol } G = 1$)

$$\begin{aligned} \|u\|^2 &= \|\langle u, u \rangle\|_A = \left\| \int_G x^{-1} (\langle T_x u, T_x u \rangle') dx \right\|_A \leq \\ &\leq \left(\sup_{x \in G} \|T_x u\|' \right)^2 \leq N^2 (\|u\|')^2. \end{aligned}$$

On the other hand, let $\langle u, u \rangle' = 1$. Then applying Lemma 2.6 and Remark 2.1, we obtain

$$\begin{aligned} \langle u, u \rangle' &= \int_G g^{-1} (\langle u, u \rangle) dg = \int_G g^{-1} (\langle T_{g^{-1}} T_g u, T_{g^{-1}} T_g u \rangle') dg \leq \\ &\leq \int_G g^{-1} (\|T_{g^{-1}}\|^2 \langle T_g u, T_g u \rangle') dg \leq \\ &\leq \int_G g^{-1} (N^2 \langle T_g u, T_g u \rangle') dg = N^2 \int_G g^{-1} (\langle T_g u, T_g u \rangle') dg = N^2 \langle u, u \rangle. \end{aligned}$$

Then $(\|u\|')^2 = \|\langle u, u \rangle'\|_A \leq N^2 \|\langle u, u \rangle\|_A = N^2 \|u\|^2$. By linearity we obtain a similar estimate for u with invertible $\langle u, u \rangle'$, while the elements of such a form are dense (see [13]) in $l_2(A)$. \square

Remark 2.8 $l_2(P)$ is a direct summand in $l_2(A)$, so the previous theorem holds for $l_2(P)$.

3 K-theory of GGA-bundles

Let us recall some general constructions of K-theory, contained in [24], and also [22].

Definition 3.1 [24, I.6.7] An additive category \mathcal{C} is called *pseudo-Abelian*, if for each object E from \mathcal{C} and each morphism $p : E \rightarrow E$, satisfying to a condition $p^2 = p$ (i. e. an idempotent) there exists the kernel $\text{Ker } p$. For an arbitrary additive category \mathcal{C} there exists *associated pseudo-Abelian category* $\tilde{\mathcal{C}}$ which is a solution of the appropriate universal problem and is defined as follows [24, I.6.10]. Objects of $\tilde{\mathcal{C}}$ are pairs (E, p) , where $E \in \text{Ob}(\mathcal{C})$ and p is a projector in E . A morphism from (E, p) to (F, q) is such a morphism $f : E \rightarrow F$ of the category \mathcal{C} , that $f \circ p = q \circ f = f$.

Definition 3.2 [24, § II.1] We call *symmetrization* of an Abelian monoid M the following Abelian group $S(M)$. Consider the product $M \times M$ and its quotient monoid with respect to the equivalence relation

$$(m, n) \sim (m', n') \Leftrightarrow \exists p, q : (m, n) + (p, p) = (m', n') + (q, q).$$

This quotient monoid is a group denoted $S(M)$. If we consider now an additive category \mathcal{C} and denote through \dot{E} the isomorphism class of an object E from \mathcal{C} , then the set $\Phi(\mathcal{C})$ of these classes is equipped with a structure of an Abelian monoid with respect to operation $\dot{E} + \dot{F} = (E \oplus F)^\bullet$. In this case the group $S(\Phi(\mathcal{C}))$ is denoted through $K(\mathcal{C})$ and is called *Grothendieck group* of the category \mathcal{C} .

Definition 3.3 [24, § II.2] *Banach structure* on an additive category \mathcal{C} is defined by the introducing of the structure of a Banach space on all groups $\mathcal{C}(E, F)$, where E and F are arbitrary objects from \mathcal{C} . It is assumed, that applications of composition of morphisms $\mathcal{C}(E, F) \times \mathcal{C}(F, G) \rightarrow \mathcal{C}(E, G)$ are bilinear and continuous. In this case we call \mathcal{C} *Banach category*.

Definition 3.4 [24, § II.2] Let \mathcal{C} and \mathcal{C}' be additive categories. An additive functor $\varphi : \mathcal{C} \rightarrow \mathcal{C}'$ is called *quasi-surjective* if each object of \mathcal{C}' is a direct summand of an object of type $\varphi(E)$. A functor φ called *full* if for any $E, F \in \text{Ob}(\mathcal{C})$ the map $\varphi(E, F) : \mathcal{C}(E, F) \rightarrow \mathcal{C}'(\varphi(E), \varphi(F))$ is surjective. For Banach categories φ is called *Banach* if this map $\varphi(E, F)$ linear and continuous.

Definition 3.5 [24, II.2.13] Let $\varphi : \mathcal{C} \rightarrow \mathcal{C}'$ be a quasi-surjective Banach functor. We shall denote by $\Gamma(\varphi)$ the set consisting of triples of the form (E, F, α) , where E and F are objects of the category \mathcal{C} and $\alpha : \varphi(E) \rightarrow \varphi(F)$ is an isomorphism. The triples (E, F, α) and (E', F', α') are named *isomorphic*, if there are such isomorphisms $f : E \rightarrow E'$ and $g : F \rightarrow F'$, that the diagram

$$\begin{array}{ccc} \varphi(E) & \xrightarrow{\alpha} & \varphi(F) \\ \varphi(f) \downarrow & & \downarrow \varphi(g) \\ \varphi(E') & \xrightarrow{\alpha'} & \varphi(F') \end{array}$$

commutes. A triple (E, F, α) is *elementary* if $E = F$ and isomorphism α is homotopic in the set of automorphisms of $\varphi(E)$ to the identical isomorphism $\text{Id}_{\varphi(E)}$. We define *sum* of two triples (E, F, α) and (E', F', α') as

$$(E \oplus E', F \oplus F', \alpha \oplus \alpha').$$

The Grothendieck group $K(\varphi)$ of a functor φ is defined as quotient set of the monoid $\Gamma(\varphi)$ with respect to the following equivalence relation: $\sigma \sim \sigma'$ if and only if there exist such elementary triples τ and τ' , that the triple $\sigma + \tau$ is isomorphic to the triple $\sigma' + \tau'$. The operation of addition introduces on $K(\varphi)$ a structure of Abelian group. The class of a triple we shall denote by $d(E, F, \alpha)$.

Definition 3.6 [24, II.3.3] Consider the set of pairs of the form (E, α) , where E is an object of the category \mathcal{C} and α is an automorphism of E . Two pairs (E, α) and (E', α') are called *isomorphic*, if there is such isomorphism $h : E \rightarrow E'$ in category \mathcal{C} , that the diagram

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \alpha \downarrow & & \downarrow \alpha' \\ E & \xrightarrow{h} & E' \end{array}$$

commutes. The direct sum defines the operation of *addition* of pairs. A pair (E, α) is called *elementary*, if the automorphism α is homotopic to Id_E in the set of automorphisms of E . Abelian group [24, II.3.4] $K^{-1}(\mathcal{C})$ is defined as a quotient set (with operation of addition) of the set of pairs $\{(E, \alpha)\}$ with respect to the following equivalence relation: $\sigma \sim \sigma'$ if and only if there are such elementary pairs τ and τ' , that $\sigma + \tau$ is isomorphic to $\sigma' + \tau'$.

Definition 3.7 [24, II.4.1] Let \mathcal{C} be a Banach category and $C^{p,q}$ be the Clifford algebra. We shall denote by $\mathcal{C}^{p,q}$ the category, objects of which are pairs (E, ρ) , where $E \in \text{Ob}(\mathcal{C})$ and $\rho : C^{p,q} \rightarrow \text{End}(E)$ is a homomorphism of algebras. A morphism from a pair (E, ρ) to a pair (E', ρ') is such a \mathcal{C} -morphism $f : E \rightarrow E'$, that $f \circ \rho(\lambda) = \rho'(\lambda) \circ f$ for each element $\lambda \in C^{p,q}$.

Definition 3.8 [24, III.4.11] Let \mathcal{C} be a pseudo-Abelian Banach category. The group $K^{p,q}(\mathcal{C})$ is defined as the Grothendieck group of the functor $\mathcal{C}^{p,q+1} \rightarrow \mathcal{C}^{p,q}$ in the sense of the Definition 3.5.

The following statement can be easily obtained by the properties of Clifford algebras.

Theorem 3.9 [24, III.4.6, III.4.12] *The groups $K^{p,q}(\mathcal{C})$ depend only on the difference $p - q$. Besides, the groups $K^{0,0}(\mathcal{C})$ and $K^{0,1}(\mathcal{C})$ are canonically isomorphic to groups $K(\mathcal{C})$ and $K^{-1}(\mathcal{C})$.*

Definition 3.10 Now we can define $K^{p-q}(\mathcal{C}) = K^{p,q}(\mathcal{C})$ and similarly for K-groups of functors.

We need also another description of K-groups, which is equivalent [24, §§ III.4, III.5] to the initial.

Definition 3.11 [24, III.4.11, III.5.1] Let \mathcal{C} be a pseudo-Abelian Banach category and let E be a $C^{p,q}$ -module (an object of the category $\mathcal{C}^{p,q}$). *Gradation* of E is such endomorphism η of object E (considering as an object from \mathcal{C}), that

1. $\eta^2 = 1$,
2. $\eta\rho(e_i) = -\rho(e_i)\eta$, where e_i are the generators of Clifford algebra and $\rho : C^{p,q} \rightarrow \text{End}(E)$ is the homomorphism, determining the $C^{p,q}$ -structure on E .

In other words, a gradation of E is a $C^{p,q+1}$ -structure on E , extending the initial $C^{p,q}$ -structure (if we put $\rho(e_{p+q+1}) = \eta$).

The term “gradation” arises from the following fact. Morphism η determines the decomposition into a direct sum

$$E = E_0 \oplus E_1, \quad E_0 = \text{Ker} \left(\frac{1-\eta}{2} \right), \quad E_1 = \text{Ker} \left(\frac{1+\eta}{2} \right),$$

while $\rho : C^{p,q} \rightarrow \text{End}(E_0 \oplus E_1)$ is a morphism of $\mathbf{Z}/2\mathbf{Z}$ -graded algebras.

Let us define group $K^{p,q}(\mathcal{C})$ as a quotient group of the free Abelian group, generated by triples (E, η_1, η_2) , where E is a $C^{p,q}$ -module and η_1, η_2 is a gradation of E , with respect to the subgroup, generated by relations

1. $(E, \eta_1, \eta_2) \oplus (F, \xi_1, \xi_2) = (E \oplus F, \eta_1 \oplus \xi_1, \eta_2 \oplus \xi_2)$,
2. $(E, \eta_1, \eta_2) = 0$, if η_1 is homotopic to η_2 in the set of gradations of E .

As usual, by $d(E, \eta_1, \eta_2) \in K^{p,q}(\mathcal{C})$ we shall denote the class of triple (E, η_1, η_2) .

We pass to the necessary specification of these constructions.

If X is a paracompact topological space, let $\text{Vect}(X; A)$ be the category of locally trivial bundles $p : E \rightarrow X$ with fiber $M = p^{-1}(x) \in \mathcal{P}(A)$ and structure group equal to $\text{Aut}_A M$. Such bundles are called *A-bundles*. This category is Banach in the sense of Definition 3.3 (see also [22, 31, 23]). Let G be a compact Lie group, acting on X and algebra A continuously.

An A -bundle $p : E \rightarrow X$ is called *GGA-bundle* if a G -space structure is given on E , and for any elements $g \in G, e \in E, x \in X$

1. $gp(e) = pg(e)$, and
2. $g : p^{-1}(x) \rightarrow p^{-1}(gx)$ is an GGA -linear mapping (i. e. $g(e \cdot a) = g(e) \cdot g(a)$).

We form the Banach category $\mathbf{Vect}_G(X; A)$ whose objects are GGA -bundles and whose morphisms are the morphisms of $\text{Vect}(X; A)$ that commute with the action of G .

The set of all continuous sections $s : X \rightarrow E$ of a GGA -bundle E over a compact space X forms a Banach A -module $\Gamma(E)$ (in the topology of the maximum of the norm). The group G strongly continuously acts on $\Gamma(E)$ according to the rule $(Gs)(x) = gs(g^{-1}x)$, where $g \in G, s \in \Gamma(E)$, and $x \in X$. Let us note that this module is a GGA -module over the algebra of all continuous A -valued functions on X :

$$\begin{aligned} g(s \cdot a)(x) &= g((s \cdot a)(g^{-1}x)) = g(s(g^{-1}x) \cdot a(g^{-1}x)) = \\ &= g(s(g^{-1}x)) \cdot g(a(g^{-1}x)) = g(s)(x) \cdot g(a)(x) = \\ &= (g(s) \cdot g(a))(x). \end{aligned}$$

The averaging mapping $s \mapsto \int_G gs$ determines a projection $\mu : \Gamma(E) \rightarrow \Gamma^G(E)$, where $\Gamma^G(E)$ is the space of G -invariant sections.

Lemma 3.12 *Let s' be a G -invariant cross-section of a GGA -bundle $E \rightarrow X$ over a closed G -stable subset¹ Y of the compact G -space X . Then s' can be extended to a G -invariant cross-section over X .*

¹That is $GY \subset Y$.

Proof: Similarly to [1, 2.1.1]. \square

Consider two *GGA*-bundles E and F over the compact base X . The G -bundle $\text{Hom}(E, F)$ is introduced in the standard way. It is not an A -bundle, but Lemma 3.12 remains valid for it. As usual, we can identify $\Gamma^G(\text{Hom}(E, F))$ with the set of *GGA*-morphisms $\varphi : E \rightarrow F$. Using Lemma 3.12, we obtain the following assertion.

Lemma 3.13 *Let $\varphi' : E|_Y \rightarrow F|_Y$ be a morphism of the restrictions of *GGA*-bundles E and F over a compact space X to a closed G -stable subset Y . Then φ' can be extended to a morphism of G - A -bundles $\varphi : E \rightarrow F$ over X . If φ' is an isomorphism, then there exists a G -stable open neighborhood U of Y such that*

$$\varphi|_U : E|_U \rightarrow F|_U$$

is an isomorphism. Any two such extensions φ_0 and φ_1 are homotopic to each other over some G -neighborhood $U' \supset Y$ in the class of isomorphisms. \square

If E is a G -space and I is the unit interval, then we define the action of G on $Y \times I$ by the formula $g(y, t) = (gy, t)$. From Lemma 3.13, as well as in a classical case, we obtain the following fact.

Lemma 3.14 *Let Y be a compact G -space, $f_t : Y \rightarrow X$ a homotopy of G -mappings ($0 \leq t \leq 1$), and E a G - A -bundle over X . Then $f_0^*E \cong f_1^*E$.* \square

By Lemma 3.13 we can define *GGA*-bundle

$$E_1 \cup_\varphi E_2 \rightarrow X$$

in the usual way, where $X = X_1 \cup X_2$, $Y = X_1 \cap X_2$, X_1 and X_2 are closed G -subspaces of the compact space X , $E_1 \rightarrow X_1$ and $E_2 \rightarrow X_2$ are *GGA*-bundles, and $\varphi : E_1|_Y \rightarrow E_2|_Y$ is an isomorphism. Further, as follows from Lemma 3.14, $E_1 \cup_\varphi E_2$ up to isomorphism depends only on the G -homotopy class of φ .

Theorem 3.15 (see, e. g. [39]) *Let E and F be A -bundles over X , and $\alpha : E \rightarrow F$ a morphism such that $\alpha_x : E_x \rightarrow F_x$ is an epimorphism for all points $x \in X$. Then there exists a morphism $\beta : F \rightarrow E$, such that $\alpha\beta = \text{Id}_F$.*

Definition 3.16 Consider a strongly continuous action of G on an arbitrary Banach space Γ . A vector $s \in \Gamma$ is called *periodic*, if the orbit Gs lies in a finite-dimensional subspace of space Γ .

According to a lemma of Mostow (see [32]) periodic vectors form a dense subset of Γ .

Theorem 3.17 *Let X be a compact G -space, and $E \rightarrow X$ a *GGA*-bundle. Then there exists a trivial *GGA*-bundle $\underline{M} = X \times M$ and *GGA*-bundle E' such that $\underline{M} \cong E \oplus E'$, where M is a projective *GGA*-module.*

Proof: By a lemma of Mostow [32] and Lemma [31] there exist sections $\bar{s}_1, \dots, \bar{s}_n$ periodic in $\Gamma(E)$, as in complex Banach space, with $\{\bar{s}_j(x)\}$ generating $E|_x$ for any x . Taking their orbits, we consider finite-dimensional G - \mathbf{C} -module $W \subset \Gamma(E)$ spanned by them: W is the \mathbf{C} -linear span of $\{G\bar{s}_j\}_{j=1, \dots, n}$. Let s_1, \dots, s_N be a basis for W . We have the trivial GGA -bundle

$$\underline{M} = X \times M = X \times (A \otimes_{\mathbf{C}} W)$$

with the diagonal action of G . Define a morphism $\theta : \underline{M} \rightarrow E$ on the generators by the formula $\theta(x, a \otimes s_i) = s_i(x) \cdot a$. This is an epimorphism of A -bundles. Since

$$\begin{aligned} \theta(g(x, a \otimes s_i)) &= \theta(gx, ga \otimes g(s_i)) = (g(s_i))(gx) \cdot ga = \\ &= gs_i(g_{-1}gx) \cdot ga = g(s_i(x)) \cdot ga = g(s_i(x) \cdot a) = g(\theta(x, a \otimes s_i)), \end{aligned}$$

it follows that θ is G -mapping. We introduce in \underline{M} the following fiberwise A -Hermitian product. If $(,)$ is a G -invariant inner product in W , then let

$$\langle (x, a \otimes s_i), (x, b \otimes s_j) \rangle = a^*b(s_i, s_j).$$

Then

$$\begin{aligned} \langle g(x, a \otimes s_i), g(x, b \otimes s_j) \rangle &= g(a)^*g(b)(g(s_i), g(s_j)) = g(a^*b)(s_i, s_j) = \\ &= g(\langle (x, a \otimes s_i), (x, b \otimes s_j) \rangle). \end{aligned}$$

Hence, our A -product is G -invariant (in the sense of Def. 2.2). Such a product gives, in particular, the structure of a Hilbert module in any fiber. Let $E' = \text{Ker } \theta$. With the help of [30] it is easy to obtain that $\underline{M} \cong E' \oplus E$. \square

Corollary 3.18 *A fiberwise GGA -product can be introduced on every GGA -bundle over a compact base.*

Let $E \rightarrow X$ be an A -bundle. We consider the bundle $\text{Hom}(E, E)$ with fiber $\text{Hom}(E_x, E_x)$, $x \in X$. An element $p_x = p_x^2$ is called a *projection in the fiber*. Denote by $\text{Proj}(E) \subset \text{Hom}(E, E)$ the set of projections. Let $Q(E) \subset \text{Hom}(E, E)$ consist of all T such that $z \cdot 1_{E_x} - T_x$ is an isomorphism for any $x \in X$, $z \in \mathbf{C}$ if $\text{Re } z \neq 1/2$. It is clear, that $\text{Proj}(E) \subset Q(E)$. The usual method of Cauchy integrals (cf. [22, p. 184]) can be used to prove

Lemma 3.19 *There exists a retraction $\alpha : Q(E) \rightarrow \text{Proj}(E)$.* \square

Construction 3.20 If E' is a GGA -bundle over a closed G -invariant subspace Y of a compact G -space X , then $E' \oplus F' = Y \times M$ for some F' , by Theorem 3.17. We define $p' : Y \times M \rightarrow Y \times M$ by the formula $p'(e', f') = (e', 0)$, bearing in mind the identifications indicated above. Then p' is a G -projection with $\text{Im } p' = E'$. We extend p' as a cross-section in $\Gamma^G(\text{Hom}(E, F))$ over Y to a G -section p over X , where $F = X \times M$. By Lemma on generators from [31], there exists a neighborhood U of Y such that $p|_{\overline{U}} \in Q(F|_{\overline{U}})$. Taking GU , we may assume that U is G -stable. It is easy to see that $E = \text{Im } \alpha(p|_{\overline{U}})$ is a GGA -bundle and $E|_Y = E'$.

Definition 3.21 Following the general scheme presented above, we define the K -groups for a compact G -space X by setting

$$\begin{aligned}\mathbf{K}_G^{p,q}(X, A) &= K^{p,q}(\mathbf{Vect}_G(X, A)), \\ \mathbf{K}_G^n(X, A) &= K^n(\mathbf{Vect}_G(X, A)), \\ \mathbf{K}_G(X, A) &= K^{0,0}(\mathbf{Vect}_G(X, A)).\end{aligned}$$

Let $\rho^{X,Y} : \mathbf{Vect}_G(X, A) \rightarrow \mathbf{Vect}_G(Y, A)$ be the restriction functor, where Y is a closed G -subspace of X . The following assertion is a consequence of Theorem 3.17 and Lemma 3.13.

Lemma 3.22 $\rho^{X,Y}$ is a full quasi surjective Banach functor in the sense of 3.4 . \square

Therefore, setting $\mathbf{K}_G^n(X, Y, A) := \mathbf{K}^n(\rho^{X,Y})$ in the sense of Definition 3.10, we get (by [24, II.3.22] and [22, 2.3.1]) the exact sequence of a pair

$$\begin{aligned}\dots \rightarrow \mathbf{K}_G^{n-1}(X, Y, A) \rightarrow \mathbf{K}_G^{n-1}(X, A) \rightarrow \mathbf{K}_G^{n-1}(Y, A) \rightarrow \\ \rightarrow \mathbf{K}_G^n(X, Y, A) \rightarrow \mathbf{K}_G^n(X, A) \rightarrow \mathbf{K}_G^n(Y, A) \rightarrow \dots\end{aligned}\tag{2}$$

Consider the trivial G -pair (B^n, S^{n-1}) , where B^n is the n -dimensional closed ball, and S^{n-1} its boundary. Then, according to a construction in [24, 22] for an arbitrary Banach category, the categories $\mathbf{Vect}_G(X, A)(B^n)$, $\mathbf{Vect}_G(X, A)(S^{n-1})$ are defined along with the group $\mathbf{K}(B^n, S^{n-1}; \mathbf{Vect}_G(X, A)) = K(\varphi)$, where

$$\varphi : \mathbf{Vect}_G(X, A)(B^n) \rightarrow \mathbf{Vect}_G(X, A)(S^{n-1})$$

is the restriction functor.

We now give the definitions and show, that φ coincides with

$$\rho^{B^n \times X, S^{n-1} \times X} : \mathbf{Vect}_G(B^n \times X, A) \rightarrow \mathbf{Vect}_G(S^{n-1} \times X, A).$$

Since both functors are induced by restrictions, it suffices to show, that for a G -trivial compact space Z the categories $\mathbf{Vect}_G(Z \times X, A)$ and $\mathbf{Vect}_G(X, A)(Z)$ are naturally isomorphic to each other. By definition, $\mathbf{Vect}_G(X, A)(Z)$ is associated in the sense of 3.1 with the category $\mathbf{Vect}_G(X, A)_T(Z)$, defined as follows. The objects of $\mathbf{Vect}_G(X, A)_T(Z)$ coincide with those of $\mathbf{Vect}_G(X, A)$, and the morphisms are continuous mappings

$$Z \rightarrow \text{Mor } \mathbf{Vect}_G(X, A)(E, F).$$

Thus, $\mathbf{Vect}_G(X, A)_T(Z)$ is identified with a full subcategory in $\mathbf{Vect}_G(X \times Z, A)$. The category

$$\mathbf{Vect}_G^T(X \times Z, A)$$

of trivial GGA -bundles over $X \times Z$, with which $\mathbf{Vect}_G(X \times Z, A)$ is associated according to Theorem 3.17 is a full subcategory of $\mathbf{Vect}_G(X, A)_T(Z)$. Passing to the associated categories, we get the required result. Comparison of it with constructions in [24, 22], yields two important corollaries.

Theorem 3.23 (Bott-Clifford periodicity) *We have a natural isomorphism*

$$\mathbf{K}_G^n(X, Y, A) \cong \mathbf{K}_G^{n-2}(X, Y, A).$$

Theorem 3.24 (Bott periodicity) *We have a natural isomorphism*

$$\mathbf{K}_G^1(X, A) \cong \mathbf{K}_G(X \times B^1, X \times S^0, A).$$

We get the next result from Construction 3.20.

Lemma 3.25 *Let Y be a closed G -subspace of a compact space X . Then*

$$\mathbf{K}_G(Y, A) = \varinjlim_{Y \subseteq U} \mathbf{K}_G(U, A).$$

Definition 3.26 Let X be a locally compact paracompact (Hausdorff) G -space. Let

$$\mathbf{K}_{G,c}(X, A) := \text{Ker} \{ \mathbf{K}_G(\dot{X}, A) \rightarrow \mathbf{K}_G(pt, A) \},$$

where $\dot{X} = X \cup pt$ is the one-point compactification.

It is easy to get another description of groups $\mathbf{K}_{G,c}$ from a consideration of the sequence (2) for the pair (\dot{X}, pt) :

Lemma 3.27

$$\mathbf{K}_{G,c}(X, A) \cong \mathbf{K}_G(\dot{X}, pt, A).$$

Definition 3.28 Let

$$\mathbf{K}_{G,c}^{-1}(X, A) = \text{Ker} \{ \mathbf{K}_G^{-1}(\dot{X}, A) \rightarrow \mathbf{K}_G^{-1}(pt, A) \},$$

$$\mathbf{K}_{G,c}^n(X, A) = \mathbf{K}_{G,c}((X \setminus Y) \times \mathbf{R}^n, A).$$

It is necessary to verify compatibility:

$$\mathbf{K}_{G,c}^{-1}(Y, A) \cong \mathbf{K}_{G,c}(Y \times \mathbf{R}, A).$$

This is done in the same way as for Theorem [24, II.4.8]. If X and Y are compact, then $\mathbf{K}_{G,c}^n(X, Y, A) \cong \mathbf{K}_G^n(X, Y, A)$; therefore, we omit the sign c .

We finish this Section with another description of our \mathbf{K} -groups.

Definition 3.29 Let X be a locally compact paracompact G -space. A *complex of GGA-bundles* over X is defined to be a sequence

$$(E, d) = \left(\dots \xrightarrow{d_i} E^i \xrightarrow{d_{i+1}} E^{i+1} \xrightarrow{d_{i+2}} \dots \right), \quad i \in \mathbf{Z},$$

where $i \in \mathbf{Z}$, the E^i are GGA -bundles over X , and d_i are morphisms with $d_{i+1}d_i = 0$ for every i ; also, $E^i = 0$ for all but perhaps finitely many i . A *morphism of complexes* $f : (E, d) \rightarrow (F, h)$ is defined to be a sequence of morphisms $f_i : E^i \rightarrow F^i$ connected by the condition $f_{i+1}d_{i+1} = h_{i+1}f_i$. Isomorphism in this category will be denoted by $(E, d) \cong (F, h)$. A point $x \in X$ is called a *point of acyclicity* (E, d) if the restriction of (E, d) to x , i. e., the sequence of A -modules

$$(E, d)_x = \left(\dots \xrightarrow{(d_i)_x} E_x^i \xrightarrow{(d_{i+1})_x} E_x^{i+1} \xrightarrow{(d_{i+2})_x} \dots \right),$$

is exact. The *support* $\text{supp}(E, d)$ is the complement in X of the set of points of acyclicity.

Proposition 3.30 $\text{supp}(E, d)$ is a closed G -subset of X .

Proof: The G -invariance is obvious. The proof of the fact that it is closed can be found, e. g., in [39, 1.3.34]. \square

Remark 3.31 The remaining assertions of this section can be proved by the classical scheme (cf. [1, 16]) with the specific nature of C^* -algebras taken into account, as it was demonstrated in two previous lemmas; therefore, the proofs are omitted.

Definition 3.32 Let Y be a closed G -subspace of X . Denote by $\mathbf{L}_G(X, Y; A)$ the semigroup (with respect to the direct sum) of classes of isomorphic GGA -complexes (E, d) on X such that $\text{supp}(E, d)$ is a compact subset of $X \setminus Y$. Two elements in $\mathbf{L}_G(X, Y; A)$ are said to be *homotopic* if for representatives (E_0, d_0) and (E_1, d_1) of them there is a complex (E, d) representing an element of $\mathbf{L}_G(X \times I, Y \times I; A)$ such that $(E_0, d_0) = (E, d)|_{X \times \{0\}}$ and $(E_1, d_1) = (E, d)|_{X \times \{1\}}$; in this case we write $(E_0, d_0) \simeq (E_1, d_1)$. We introduce the following equivalence relation: $(E_0, d_0) \sim (E_1, d_1)$ iff there are acyclic (F_0, f_0) and (F_1, f_1) such that

$$(E_0, d_0) \oplus (F_0, f_0) \simeq (E_1, d_1) \oplus (F_1, f_1).$$

By acyclicity we understand the condition $\text{supp}(E, d) = \emptyset$. Let $\mathbf{M}_G(X, Y; A) = \mathbf{L}_G(X, Y; A) / \sim$. Denote by $\mathbf{L}_G^n(X, Y; A)$ and $\mathbf{M}_G^n(X, Y; A)$ the corresponding semigroups constructed from the complexes of length n . We have the natural injective semigroup homomorphisms (addition of the zero term):

$$\mathbf{L}_G^n(X, Y; A) \rightarrow \mathbf{L}_G^{n+1}(X, Y; A), \quad \mathbf{L}_G(X, Y; A) := \varinjlim \mathbf{L}_G^n(X, Y; A).$$

The equivalence relation \sim commutes with imbeddings; therefore, the indicated morphisms induce morphisms $\mathbf{M}_G^n(X, Y; A) \rightarrow \mathbf{M}_G^{n+1}(X, Y; A)$.

Lemma 3.33 Suppose that $E \rightarrow X$ and $F \rightarrow X$ are GGA -bundles, $\alpha : E|_Y \rightarrow F|_Y$ and $\beta : E \rightarrow F$ are monomorphisms, and in the class of monomorphisms $E|_Y \rightarrow F|_Y$ there exists a G -homotopy joining α and $\beta|_Y$. Then there exists a monomorphism $\tilde{\alpha} : E \rightarrow F$ such that $\tilde{\alpha}|_Y = \alpha$. \square

Lemma 3.34 $\mathbf{M}_G^n(X, Y; A) \rightarrow \mathbf{M}_G^{n+1}(X, Y; A)$ is an isomorphism. \square

Remark 3.35 Suppose that X is compact and $Y = \emptyset$. Then we have a natural isomorphism $\chi_1 : \mathbf{M}_G^1(X, \emptyset; A) \rightarrow \mathbf{K}_G(X; A)$, assigning to a class of the complex $(0 \rightarrow E^0 \xrightarrow{\alpha} E^1 \rightarrow 0)$ the element $[E^1] - [E^0]$.

Lemma 3.36 There exists a unique natural equivalence of functors

$$\chi_1 : \mathbf{M}_G^1(X, Y; A) \rightarrow \mathbf{K}_G(X, Y; A),$$

on the category of compact G -pairs, with the equivalence of the form indicated in 3.35 for (X, \emptyset) .

Definition 3.37 We return to the case of (locally compact, paracompact) noncompact, in general, G -pairs (X, Y) . Two complexes (E, d) and (F, b) in $\mathbf{L}_G(X, Y; A)$ are said to be *compactly isomorphic* if there exists a compact G -subset C with

$$\text{supp}(E, d) \cup \text{supp}(F, b) \subset \text{int } C \subset C \subset X \setminus Y,$$

and G -isomorphisms $\psi^i : E^i \rightarrow F^i$ over X , where the following diagram commutes over $X \setminus \text{int } C$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E^0 & \xrightarrow{d^1} & E^1 & \xrightarrow{d^2} & \dots & \xrightarrow{d^n} & E^n & \longrightarrow & 0 \\ & & \downarrow \psi^0 & & \downarrow \psi^1 & & & & \downarrow \psi^n & & \\ 0 & \longrightarrow & F^0 & \xrightarrow{b^1} & F^1 & \xrightarrow{b^2} & \dots & \xrightarrow{b^n} & F^n & \longrightarrow & 0. \end{array}$$

Complex of the form $0 \rightarrow \dots \rightarrow 0 \rightarrow E \xrightarrow{\text{Id}} E \rightarrow 0 \rightarrow \dots \rightarrow 0$ is said to be *elementary*. Two complexes (E, d) and (F, b) are said to be *compact equivalent*, if there exist elementary Q_1, \dots, Q_r and P_1, \dots, P_s such that $(E, d) \oplus Q_1 \oplus \dots \oplus Q_r$ is compact isomorphic to $(F, b) \oplus P_1 \oplus \dots \oplus P_s$. This equivalence is denoted by \approx .

Lemma 3.38 $(E, d) \approx (F, b)$ implies $(E, d) \sim (F, b)$. □

Lemma 3.39 Suppose that

$$(E, d) = (0 \rightarrow E^0 \xrightarrow{d} E^1 \rightarrow 0), \quad (F, b) = (0 \rightarrow F^0 \xrightarrow{b} F^1 \rightarrow 0)$$

and there exist compact G -sets C_1 and C_2 such that

$$\text{supp}(E, d) \cup \text{supp}(F, b) \subset C_1 \subset \text{int } C_2 \subset C_2 \subset X \setminus Y,$$

and a GGA-bundle L over C_2 with an isomorphism

$$\theta : (E^0 \oplus F^1)|_{C_2} \oplus L \rightarrow (E^1 \oplus F^0)|_{C_2} \oplus L,$$

for which $\theta|_{C_2 \setminus C_1} = d \oplus b^{-1} \oplus 1$. Then $(E, d) \approx (F, b)$. □

Lemma 3.40 $\mathbf{M}_G^1(X \setminus Y; A) \cong \mathbf{M}_G^1(X, Y; A)$. □

Corollary 3.41

$$\begin{array}{ll} 1) & \mathbf{M}_G^1(\dot{X}, pt; A) \cong \mathbf{M}_G^1(X; A) \\ & \downarrow \cong \\ & \mathbf{K}_G(\dot{X}, pt; A) \cong \mathbf{K}_G(X; A), \\ 2) & \mathbf{M}_G^1(X, Y; A) \cong \mathbf{M}_G^1(X \setminus Y; A) \\ & \downarrow \cong \\ & \mathbf{K}_G(X, Y; A) \cong \mathbf{K}_G(X \setminus Y; A) \end{array}$$

□

Theorem 3.42 There is a natural isomorphism

$$\mathbf{M}_G(X, Y; A) \cong \mathbf{M}_G^1(X, Y; A) \cong \mathbf{K}_G(X, Y; A). \quad \square$$

4 *-Fredholm operators

Let us remind the definition of C*-Fredholm operator [31, 41].

Definition 4.1 A bounded A -operator $F : H_A \rightarrow H_A$, is called *Fredholm*, if

- 1) The operator F supposes adjoint, and
- 2) There exist decompositions of the domain $H_A = M_1 \oplus N_1$ and the values space $H_A = M_2 \oplus N_2$ of F (where M_1, M_2, N_1, N_2 are closed A -submodules, N_1 and N_2 have a finite number of generators) such that the operator F has with respect to these decompositions the matrix $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$, and $F_1 : M_1 \rightarrow M_2$ is an isomorphism.

In the equivariant case we change the definition as follows. Let $l_2(P)$ be equipped with an invariant A -inner product (it is possible to do this by Remark 2.8). We can apply the Stabilization Theorem of Kasparov [26] to $l_2(P)$:

$$l_2(P) \oplus \mathcal{H}_A \cong \mathcal{H}_A, \quad (3)$$

where $\mathcal{H}_A = \sum_{i=1}^{\infty} (A \otimes_{\mathbf{C}} V_i)$, and $\{V_i\}$ is the countable collection of finite-dimensional spaces, in which all (up to isomorphism) irreducible unitary representations of G are realized, and each representation is repeated an infinite number of times. Isomorphism (3) is a GGA -isomorphism of Hilbert modules and the sum on the left in (3) is orthogonal. We introduce the following notation

$$R_m = \sum_{i=m+1}^{\infty} (A \otimes_{\mathbf{C}} V_i), \quad R_m^{\perp} = \sum_{i=1}^m (A \otimes_{\mathbf{C}} V_i).$$

For any bounded G - A -operator $F : l_2(P_1) \rightarrow l_2(P_2)$ let $S(F) : \mathcal{H}_A \rightarrow \mathcal{H}_A$ (S due to the word “stabilization”) denote the operator

$$\begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix} : \mathcal{H}_A \cong \mathcal{H}_A \oplus l_2(P_1) \rightarrow \mathcal{H}_A \oplus l_2(P_2) \cong \mathcal{H}_A.$$

Of course, everything is determined up to a GGA -isomorphism.

Theorem 4.2 cf. [31, 41] *Let $\mathcal{H}_A \cong \mathcal{M} \oplus \mathcal{N}$, where \mathcal{M} and \mathcal{N} are closed GGA -modules, and \mathcal{N} has a finitely many generators a_1, \dots, a_s . Then \mathcal{N} is a projective GGA -module of finite type.*

Proof: Just as in [31, 41]. □

Definition 4.3 A bounded GGA -operator

$$F : l_2(\mathcal{P}_1) \rightarrow l_2(\mathcal{P}_2),$$

is called *Fredholm operator* (GGA -Fredholm), if

- 1) F admits an adjoint;
- 2) for $S(F)$ there exist an inverse image decomposition $\mathcal{H}_A = \mathcal{M}_1 \oplus \mathcal{N}_1$ and an image decomposition $\mathcal{H}_A = \mathcal{M}_2 \oplus \mathcal{N}_2$, where $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ are closed GGA -modules, $\mathcal{N}_1, \mathcal{N}_2$ have finitely many generators, and the operator $S(F)$ has the matrix form $S(F) =$

$\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ in these decompositions, with $F_1 : M_1 \rightarrow M_2$ being an isomorphism of GGA -modules. By Theorem 4.2, N_1 and N_2 are projective GGA -modules; we can form the index element

$$\text{index } F = [\mathcal{N}_1] - [\mathcal{N}_2] \in \mathbf{K}^G(A).$$

Theorem 4.4 (see, e. g. [39]) *In the decomposition in the definition of A -Fredholm operator (see 4.1) we can always assume M_0 and M_1 admitting an orthogonal complement. More precisely, there exists a decomposition for F*

$$\begin{pmatrix} F_3 & 0 \\ 0 & F_4 \end{pmatrix} : H_A = V_0 \oplus W_0 \rightarrow V_1 \oplus W_1 = H_A,$$

such that $V_0^\perp \hat{\oplus} V_0 = H_A$, $V_1^\perp \hat{\oplus} V_1 = H_A$, or (what is just the same) such that the projections

$$p_0 : V_0 \oplus W_0 \rightarrow V_1, \quad p_1 : V_1 \oplus W_1 \rightarrow V_1$$

admit conjugates.

Lemma 4.5 *Let $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ be a decomposition into an orthogonal sum and \mathcal{M} and \mathcal{M}_0 are GGA -modules with an invariant inner product. Then \mathcal{M}_1 is invariant.*

Proof: Let $x \in \mathcal{M}_0$, $y \in \mathcal{M}_1$ and $g \in G$ be arbitrary. Then

$$\langle x, gy \rangle = g(\langle g^{-1}x, y \rangle) = g(0) = 0. \quad \square$$

Corollary 4.6 *The previous Theorem is valid in the GGA -case as well.*

Proof: While proving the mentioned theorem we took $V_0 = (N_0)^\perp$ and $V_1 = F(V_0)$. Hence, by Lemma all modules are invariant. \square

The remaining statements of the present section can be proved similarly to the GA -case (see [41, 39]).

Theorem 4.7 *The index is well-defined.*

Lemma 4.8 *Let $F : l_2(A) \rightarrow l_2(A)$ be a GGA -Fredholm operator, then there exists an $\varepsilon > 0$ such that any bounded GGA -operator D , restricted to satisfy $\|F - D\| < \varepsilon$ and admitting an adjoint, is a GGA -Fredholm operator and $\text{index } D = \text{index } F$.*

Lemma 4.9 *Let F and D be GGA -Fredholm operators,*

$$F : l_2(\mathcal{P}_1) \rightarrow l_2(\mathcal{P}_2), \quad D : l_2(\mathcal{P}_2) \rightarrow l_2(\mathcal{P}_3).$$

Then $DF : l_2(\mathcal{P}_1) \rightarrow l_2(\mathcal{P}_3)$ is a GGA -Fredholm operator and $\text{index } DF = \text{index } D + \text{index } F$.

Lemma 4.10 *Let $K : l_2(\mathcal{P}) \rightarrow l_2(\mathcal{P})$ be a compact GGA-operator. Then $1 + K$ is a GGA-Fredholm operator and $\text{index}(1 + K) = 0$.*

Lemma 4.11 *Consider a GGA-Fredholm operator $F : l_2(\mathcal{P}_1) \rightarrow l_2(\mathcal{P}_2)$. Let an operator $K \in \mathcal{K}(l_2(\mathcal{P}_1), l_2(\mathcal{P}_2))$ be G -equivariant. Then the operator $F + K$ is a GGA-Fredholm operator and $\text{index}(F + K) = \text{index } F$.*

Lemma 4.12 *Let $F : l_2(\mathcal{P}_1) \rightarrow l_2(\mathcal{P}_2)$ be a bounded GGA-operator admitting an adjoint, $D = S(F) + K \in \text{End}^* \mathcal{H}_A$ and $K \in \mathcal{K}(\mathcal{H}_A)$ be GGA-operators. Let D have a decomposition of \mathcal{H}_A from the definition of G -A-Fredholm operator. Then F is a GGA-Fredholm operator.*

Theorem 4.13 *Let*

$$F : l_2(\mathcal{P}_1) \rightarrow l_2(\mathcal{P}_2), \quad D : l_2(\mathcal{P}_2) \rightarrow l_2(\mathcal{P}_1), \quad D' : l_2(\mathcal{P}_2) \rightarrow l_2(\mathcal{P}_1)$$

be bounded GGA-operators admitting an adjoint and

$$S(FD) = 1_{\mathcal{H}_A} + K_1, \quad S(D'F) = 1_{\mathcal{H}_A} + K_2, \quad K_1, K_2 \in \mathcal{K}(\mathcal{H}_A).$$

Then F is a GGA-Fredholm operator.

Lemma 4.14 *Let D , D' , and F be bounded GGA-operators admitting adjoint. Let FD and $D'F$ be GGA-Fredholm operators. Then F is a GGA-Fredholm operator.*

5 The Thom isomorphism

In this section we discuss the theorem on the Thom isomorphism in $\mathbf{K}_G(\cdot, A)$ -theory. As in other cases (see e.g. [14]) it plays an important role.

Let X be a G -space, $p : F \rightarrow X$ a complex bundle over X , and $s : X \rightarrow F$ an invariant section. We denote by $\Lambda^i(F)$ the complex G -bundle of i -vectors over X . Let us define the complex $\Lambda(F, s)$ over X of lengths $n = \dim F$:

$$\Lambda(F, s) := (0 \rightarrow \Lambda^0(F) \xrightarrow{\alpha^0} \Lambda^1(F) \xrightarrow{\alpha^1} \dots \xrightarrow{\alpha^{n-1}} \Lambda^n(F) \rightarrow 0),$$

where $\alpha^k(v_x) = s(x) \wedge v_x$ for $v_x \in \Lambda^k(F)_x$.

Lemma 5.1 (see [16])

1. $(\Lambda(F, s), \alpha)$ is really a complex.
2. $\text{supp}(\Lambda(F, s)) = \{x \in X | s(x) = 0\}$ and if this set is compact, then the element $[\Lambda(F, s)] \in K_G(X)$ is defined. \square

Let $\pi : p^*F \rightarrow F$ be the bundle with total space

$$p^*F = \{(f_1, f_2) \in F \times F \mid p(f_1) = p(f_2)\}, \quad \pi(f_1, f_2) = f_1.$$

The vector bundle (p^*F, π, F) has the canonical section

$$s_F : F \rightarrow p^*F, \quad s_F(f) = (f, f).$$

The support of s_F is equal to X . Hence, if X is a compact set, the element $[\Lambda(p^*F, s_F)] = \lambda_F \in K_G(X)$ is defined. Let us define by $a \cdot b$ the element $p^*(a) \otimes b \in \mathbf{K}_G(F; A)$.

Definition 5.2 If the base of a vector bundle is compact, then the mapping

$$\varphi : \mathbf{K}_G(X; A) \rightarrow \mathbf{K}_G(F; A), \quad \varphi(a) = a \cdot \lambda_F.$$

is called the *Thom homomorphism*.

The following statement is obvious.

Proposition 5.3 *The Thom homomorphism φ is a morphism of $R(G)$ -modules.* \square

Let $i : X \hookrightarrow F$ be the enclosure of the zero section of F . It induces the homomorphisms

$$i^* : \mathbf{K}_G(F; A) \rightarrow \mathbf{K}_G(X; A), \quad i^*\varphi : K_G(X; A) \rightarrow K_G(X; A).$$

Proposition 5.4 1. *If X is compact, then the following sequence is defined*

$$0 \longrightarrow \Lambda^0 F \xrightarrow{0} \Lambda^1 F \xrightarrow{0} \Lambda^2 F \longrightarrow \dots \xrightarrow{0} \Lambda^n F \longrightarrow 0.$$

It defines an element of $K_G(X)$. For any element $a \in \mathbf{K}_G(X; A)$

$$i^*\varphi(a) = a \cdot [0 \longrightarrow \Lambda^0 F \xrightarrow{0} \dots \xrightarrow{0} \Lambda^n F \longrightarrow 0].$$

2. *For $a \in \mathbf{K}_G(X; A) \cong \mathbf{M}_G(X; A)$ (see 3.42)*

$$i^*\varphi(a) = a \cdot \sum_{i=0}^n (-1)^i \Lambda^i F.$$

Proof: By the construction of the natural isomorphism between \mathbf{K}_G and \mathbf{M}_G , these assertions are equivalent to each other. Let us prove the item 1. Let $a \in \mathbf{K}_G(X; A)$ be represented by the complex

$$a = [(\mathcal{E}, \alpha)] = [0 \longrightarrow E^k \xrightarrow{\alpha^k} \dots \xrightarrow{\alpha^1} E^0 \longrightarrow 0].$$

Then

$$\varphi(a) = a \cdot \lambda_F = [(p^*\mathcal{E}, p^*\alpha) \otimes \Lambda(p^*F, s_F)],$$

hence

$$i^*\varphi(a) = [(i^*p^*\mathcal{E}, i^*p^*\alpha) \otimes [i^*\Lambda(p^*F, s_F)]].$$

Since $pi = 1$, we have $(i^*p^*\mathcal{E}, i^*p^*\alpha) = (\mathcal{E}, \alpha) = a$, while the restriction of complex $\Lambda(p^*F, s_F)$ on X is equal to

$$[0 \longrightarrow \Lambda^0 F \xrightarrow{0} \dots \xrightarrow{0} \Lambda^n F \longrightarrow 0]. \quad \square$$

Let us pass to the case of a locally compact space X . The complex $\Lambda(p^*F, s_F)$ has no compact support now and does not determine the element $\lambda_F \in K_G(F)$. However, if $a = [(\mathcal{E}, \alpha)] \in \mathbf{K}_G(X; A)$, then

$$\begin{aligned} \text{supp } \{(p^*\mathcal{E}, p^*\alpha) \otimes \Lambda(p^*F, s_F)\} &\subset \\ &\subset \text{supp } (p^*\mathcal{E}, p^*\alpha) \cap \text{supp } \Lambda(p^*F, s_F) \subset \\ &\subset \text{supp } (p^*\mathcal{E}, p^*\alpha) \cap X = \text{supp } (\mathcal{E}, \alpha). \end{aligned}$$

Thus, the complex $(p^*\mathcal{E}, p^*\alpha) \otimes \Lambda(p^*F, s_F)$ has the compact support. We obtain a homomorphism of $R(G)$ -modules

$$\varphi : \mathbf{K}_G(X; A) \rightarrow \mathbf{K}_G(F; A), \quad \varphi(a) = [(p^*\mathcal{E}, p^*\alpha) \otimes \Lambda(p^*F, s_F)].$$

As well as in the compact case,

$$i^*\varphi(a) = a [0 \rightarrow \Lambda^0 F \xrightarrow{0} \Lambda^1 F \xrightarrow{0} \dots \xrightarrow{0} \Lambda^n F \rightarrow 0].$$

Passing by the Bott periodicity to K^1 (see Theorem 3.23), we define the *Thom homomorphism* in the general case:

$$\varphi = \varphi_A^F : \mathbf{K}_G^*(X; A) \rightarrow \mathbf{K}_G^*(F; A).$$

Let now E and F be two complex G -bundles over X . Then a product is defined by the formula

$$\mu : K_G(E) \otimes K_G(F) \rightarrow K_G(E \times F).$$

Since the enclosure $E \oplus F \rightarrow E \times F$ induces a homomorphism

$$K_G(E \times F) \rightarrow K_G(E \oplus F),$$

we obtain the multiplication $\cdot : K_G(E) \otimes K_G(F) \rightarrow K_G(E \oplus F)$.

Theorem 5.5 1. [16] *If X is compact, then*

$$\lambda_E \cdot \lambda_F = \lambda_{E \oplus F}.$$

2. *In the general case $\varphi_A^E \cdot \varphi_C^F = \varphi_C^E \cdot \varphi_A^F = \varphi_A^{E \oplus F}$. The equalities hold in the following sense. Let, for example, $ab \in \mathbf{K}_G(X; A)$, where $a \in K_G(X)$, $b \in \mathbf{K}_G(X; A)$. Then $\varphi_C^E(a) \cdot \varphi_A^F(b) = \varphi_A^{E \oplus F}(ab)$.*

3. [16] *Let F_1 and F_2 be two complex bundles over X , and s_1, s_2 their section. Then*

$$\Lambda(F_1 \oplus F_2, s_1 \oplus s_2) = \Lambda(F_1, s_1) \otimes \Lambda(F_2, s_2).$$

Proof: The item 2 immediately follows from the item 1 and the definition of the Thom homomorphism. The items 1 and 3 are proved in [16]. \square

Proposition 5.6 *Consider $E \oplus F$ as a bundle over X as well as a bundle over E . Then the diagram*

$$\begin{array}{ccc} \mathbf{K}_G^*(X; A) & \xrightarrow{\varphi^E} & \mathbf{K}_G^*(E; A) \\ \downarrow \varphi_1^{E \oplus F} & & \downarrow \varphi_2^{E \oplus F} \\ \mathbf{K}_G^*(E \oplus F; A) & = & \mathbf{K}_G^*(E \oplus F; A) \end{array}$$

is commutative.

Proof: Consider the projections

$$p : E \rightarrow X, \quad q : F \rightarrow X, \quad r : E \oplus F \rightarrow X, \quad t : E \oplus F \rightarrow E.$$

Let $x \in K_G(X; A)$. Then $\varphi^E(x) = p^*(x) \Lambda(p^*E, s_E)$,

$$\varphi_2^{E \oplus F} \varphi^E(x) = t^*(p^*(x) \Lambda(p^*E, s_E)) \Lambda(t^*(E \oplus F), s_{E \oplus F}),$$

$$t^*p^*(x) = r^*(x), \quad t^*\Lambda(p^*E, s_E) = \Lambda(r^*E, t^*s_E)$$

and $\Lambda(t^*(E \oplus F), s_{E \oplus F}) = \Lambda(r^*F, t^*s_F)$. Since $t^*s_E + t^*s_F = s_{E \oplus F}$, by the item 3 of Theorem 5.5 (we denote the elements of K-groups and their representatives by the same symbols),

$$\begin{aligned} \varphi_2^{E \oplus F} \varphi^E(x) &= r^*(x) \Lambda(r^*E, t^*s_E) \Lambda(r^*F, t^*s_F) = \\ &= r^*(x) \Lambda(r^*(E \oplus F), s_{E \oplus F}) = \varphi_1^{E \oplus F}(x). \quad \square \end{aligned}$$

Let starting from this moment X be *separable*. Then $C_0(X) \rtimes G$ is of the class of algebras for which the results of [38] are valid.

Theorem 5.7 [42] *Let X be separable and metrizable, and let action of G on A be trivial. Then φ_A is an isomorphism.*

Proof: Let us form the geometrical resolution [38]

$$0 \rightarrow A \otimes \mathcal{K} \otimes C_0(\mathbf{R}) \xrightarrow{\iota} C \xrightarrow{\nu} 0,$$

where $C \neq \mathbf{C}$ and F are C^* -algebras, and \mathcal{K} is the algebra of compact operators in Hilbert space. Let us prove the following Lemma.

Lemma 5.8 *The diagram*

$$\begin{array}{ccc} \mathbf{K}_G^*(X; A \otimes \mathcal{K} \otimes C_0(\mathbf{R})) & \xrightarrow{\iota_*} & \mathbf{K}_G^*(X; C) \\ \downarrow \varphi_{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})} & & \downarrow \varphi_C \\ \mathbf{K}_G^*(V; A \otimes \mathcal{K} \otimes C_0(\mathbf{R})) & \xrightarrow{\iota_*} & \mathbf{K}_G^*(V; C) \end{array}$$

is commutative. Here $V \rightarrow X$ is a \mathbf{C} -vector bundle,

$$\iota : A \otimes \mathcal{K} \otimes C_0(\mathbf{R}) = \text{Ker } \nu \hookrightarrow C,$$

ι_ is an extension of scalars.*

Proof: Let $a \in \mathbf{K}_G^0(X; A \otimes \mathcal{K} \otimes C_0(\mathbf{R}))$ be defined by the following complex:

$$(\mathcal{E}, \alpha) = (0 \rightarrow E^1 \xrightarrow{\alpha} E^0 \rightarrow 0).$$

Then (by $\widehat{\otimes}$ we define the tensor product over the cartesian product of bases)

$$\begin{aligned} \varphi_{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}(a) &= [(p^* \mathcal{E}, \psi^* \alpha) \otimes \Lambda(p^* F, s_F)] = \\ &= \left[0 \longrightarrow p^* E^1 \widehat{\otimes} p^* \Lambda^0 F \xrightarrow{(p^* \alpha \widehat{\otimes} 1) \oplus (1 \widehat{\otimes} s)} (p^* E^1 \widehat{\otimes} p^* \Lambda^0 F) \oplus (p^* E^1 \widehat{\otimes} p^* \Lambda^1 F) \longrightarrow \dots \right] |_{\text{diag}}, \end{aligned}$$

and

$$\begin{aligned} \iota_* \varphi_{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}(a) &= \left[0 \rightarrow (p^* E^1 \otimes_{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})} C) \widehat{\otimes} p^* \Lambda^0 F \xrightarrow{(p^* \alpha \otimes 1_C \widehat{\otimes} 1) \oplus ((1 \otimes 1_C) \widehat{\otimes} s)} \right. \\ &\rightarrow ((p^* E^0 \otimes_{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})} C) \widehat{\otimes} p^* \Lambda^0 F) \oplus ((p^* E^1 \otimes_{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})} C) \widehat{\otimes} p^* \Lambda^1 F) \rightarrow \dots \left. \right] |_{\text{diag}} = \\ &= \varphi_C \iota_*(a). \end{aligned}$$

By definition φ commutes with the Bott isomorphism, which is natural [22], in particular, it commutes with ι_* . Hence, Lemma is true for K^1 too. \square

If G is a compact group, X is a paracompact Hausdorff G -space, then X/G is a paracompact G -space (see, e. g. [16, p. 7]). The stabilizer G_x is closed [11]. Hence, it is compact and has the type I. Therefore, by the following theorem, C^* -crossed product $C_0(X) \rtimes G$ has the type I in our case.

Theorem 5.9 [18] $C_0(X) \rtimes G$ has the type I iff X/G is a τ_0 -space and all isotropy groups G_x has the type I. \square

Let $\gamma : K_G^*(X; B) \xrightarrow{\cong} K_*((C_0(X) \otimes B) \rtimes G)$ be the natural isomorphism [19, 21]. Let $B := A \otimes \mathcal{K} \otimes C_0(\mathbf{R})$ and $C_G^X := C_0(X) \rtimes G$ (and similarly for V). Consider the following diagram

$$\begin{array}{ccccccc} 0 \rightarrow \text{Tor}(K_*(C_G^X), K_*(A)) & \longrightarrow & K_*(C_G^X) \otimes K_*(C) & \longrightarrow & K_*(C_G^X \otimes C) & & \\ & \cong \downarrow \gamma \varphi_C \gamma^{-1} * 1 & \boxed{1} & \cong \downarrow \gamma \varphi_C \gamma^{-1} \otimes 1 & \boxed{2} & \downarrow \gamma \varphi_C \gamma^{-1} & \\ 0 \rightarrow \text{Tor}(K_*(C_G^V), K_*(A)) & \longrightarrow & K_*(C_G^V) \otimes K_*(C) & \longrightarrow & K_*(C_G^V \otimes C), & & \end{array}$$

where the exact rows represent a part of (4.5) from [38].

Consider also the diagram

$$\begin{array}{ccccc}
& \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^X \otimes B) & \xrightarrow{\iota_*} & \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^X \otimes C) & \\
& \uparrow \text{Bott} \cong & \boxed{8} & \cup & \uparrow \\
0 \rightarrow K_*(C_G^X) \otimes K_*(A) & \xrightarrow{\alpha_X} & K_*(C_G^X \otimes A) & \xrightarrow{\beta_X} & \text{Tor}(K_*(C_G^X), K_*(A)) \rightarrow 0 \\
& \uparrow \cong \gamma_{\otimes 1} & \boxed{3} & & \\
& K_*((C_0(X) \otimes A) \rtimes G) & & & \\
& \uparrow \gamma \cong & & & \\
0 \rightarrow K_G^*(X) \otimes K_*(A) & \longrightarrow & K_G^*(X; A) & & \downarrow \gamma \varphi \mathbf{C} \gamma^{-1} * 1 \\
& \downarrow \cong \varphi_{\mathbf{C} \otimes 1} & \boxed{5} & \downarrow \varphi_A & \\
0 \rightarrow K_G^*(V) \otimes K_*(A) & \longrightarrow & K_G^*(V; A) & & \\
& \downarrow \cong \gamma_{\otimes 1} & \boxed{4} & & \\
& K_*((C_0(V) \otimes A) \rtimes G) & & & \\
& \downarrow \gamma \cong & & & \\
& \downarrow \sigma_V \cong & & & \\
0 \rightarrow K_*(C_G^V) \otimes K_*(A) & \xrightarrow{\alpha_V} & K_*(C_G^V \otimes A) & \xrightarrow{\beta_V} & \text{Tor}(K_*(C_G^V), K_*(A)) \rightarrow 0 \\
& \downarrow \text{Bott} \cong & \boxed{9} & \cap & \downarrow \\
& \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^V \otimes B) & \xrightarrow{\iota_*} & \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^V \otimes C), &
\end{array}$$

where the rows are exact: they present the main result of [38]. Let us consider the following diagram:

$$\begin{array}{ccccccc}
& \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^X \otimes B) & \xrightarrow{\iota_*} & \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^X \otimes C) & \xleftarrow{\quad} & \text{Tor}(K_*(C_G^X), K_*(A)) \rightarrow 0 \\
& \uparrow \text{(Bott)} \sigma_X \gamma \cong & & \downarrow \gamma \varphi_B \gamma^{-1} & \boxed{10} & \downarrow \gamma \varphi_C \gamma^{-1} & \boxed{7} \\
K_G^*(X; A) & & & & & & \downarrow \gamma \varphi \mathbf{C} \gamma^{-1} * 1 \\
& \downarrow \text{(Bott)} \sigma_V \gamma \varphi_A & & \downarrow \gamma \varphi_B \gamma^{-1} & & \downarrow \gamma \varphi_C \gamma^{-1} & \\
& \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^V \otimes B) & \xrightarrow{\iota_*} & \bigoplus_{q \in \mathbf{Z}_2} K_q(C_G^V \otimes C) & \xleftarrow{\quad} & \text{Tor}(K_*(C_G^V), K_*(A)) \rightarrow 0.
\end{array}$$

Let us suppose that all squares $\boxed{1}, \dots, \boxed{10}$ are commutative. We obtain the diagram

$$\begin{array}{ccccccc}
0 \rightarrow K_G^*(X) \otimes K_*(A) & \longrightarrow & K_G^*(X; A) & \longrightarrow & \text{Tor}(K_*(C_G^X), K_*(A)) & \rightarrow 0 \\
& \cong \downarrow \varphi_{\mathbf{C} \otimes 1} & \boxed{11} & \downarrow \varphi_A & \boxed{12} & \cong \downarrow \gamma \varphi \mathbf{C} \gamma^{-1} * 1 & (4) \\
0 \rightarrow K_G^*(V) \otimes K_*(A) & \longrightarrow & K_G^*(V; A) & \longrightarrow & \text{Tor}(K_*(C_G^V), K_*(A)) & \rightarrow 0.
\end{array}$$

Its rows are exact by the commutativity of $\boxed{3}$ and $\boxed{4}$, and $\boxed{11} = \boxed{5}$. If $a \in K_G^*(X; A)$,

then

$$\begin{aligned}
 (\gamma\varphi_{\mathbf{C}}\gamma^{-1} * 1)(\beta_X \circ \sigma_X \circ \gamma(A)) &= \begin{array}{|c|c|} \hline 7 & 8 \\ \hline \end{array} \\
 = \gamma\varphi_C\gamma^{-1} \circ \iota_* \circ (\text{Bott}) \circ \sigma_X \circ \gamma(a) &= \begin{array}{|c|c|} \hline 6 & 10 \\ \hline \end{array} \\
 = \iota_* \circ (\text{Bott}) \circ \sigma_V \circ \gamma \circ \varphi_A(a) &= \begin{array}{|c|c|} \hline 7 & 9 \\ \hline \end{array} \\
 = (\beta_V \circ \sigma_v \circ \gamma)\varphi_A(a). &
 \end{aligned}$$

Hence, (4) is a commutative diagram with exact rows. Let us show that the squares $\boxed{1}, \dots, \boxed{10}$ are commutative.

- $\boxed{1}$ commutes by the standard algebraic argument;
- $\boxed{5}$ commutes by the associativity of \otimes : rows and columns are induced by \otimes ;
- $\boxed{2}$ commutes if we can prove the commutativity of $\boxed{3}$ or $\boxed{4}$ for any unital A . Indeed, C is not unital, but we can consider the unitalization C^+ of it and the diagram:

$$\begin{array}{ccccc}
 & & K_*(C_G^X) \otimes K_*(C) & \longrightarrow & K_*(C_G^X \otimes C) \\
 & \swarrow & \uparrow & \swarrow & \uparrow \\
 K_*(C_G^X) \otimes K_*(C^+) & \longrightarrow & K_*(C_G^X \otimes C^+) & & \\
 \uparrow & & \uparrow & & \uparrow \\
 K_*^G(C_0(X)) \otimes K_*(C) & \longrightarrow & K_*^G(C_0(X) \otimes C) & & \\
 \uparrow & \swarrow & \uparrow & \swarrow & \uparrow \\
 K_*^G(C_0(X)) \otimes K_*(C^+) & \longrightarrow & K_*^G(C_0(X) \otimes C^+) & &
 \end{array}$$

Then the forward square is commutative by $\boxed{3}$. The commutativity of the others is evident. Hence, the back square is commutative and we get $\boxed{2}$.

- $\boxed{8}$ and $\boxed{9}$ commute by the construction of the exact sequence from [38].
- $\boxed{6}$ commutes since γ for K_1 is defined in [19, 21] via the Bott isomorphism.
- $\boxed{7} = \boxed{1} + \boxed{2}$.

So, we have to verify the commutativity of $\boxed{3}$ (for any unital algebra) and $\boxed{10}$. Let P be a projective G - $C_0(X)$ -module, M a projective A -module. Then

$$\alpha(\gamma \otimes 1)([P] \otimes_{\mathbf{Z}} [\mathcal{M}]) = \alpha([P \otimes_{L^1(G, C_0(X))} (C_0 \rtimes G)] \otimes_{\mathbf{Z}} [\mathcal{M}]) = [(P \otimes_{L^1(G, C_0(X))} (C_0 \rtimes G)) \otimes_{\mathbf{C}} \mathcal{M}],$$

and

$$\begin{aligned}
 \sigma\gamma\alpha([P] \otimes_{\mathbf{Z}} [\mathcal{M}]) &= \sigma\gamma([P \otimes_{\mathbf{C}} \mathcal{M}]) = \sigma[(P \otimes_{\mathbf{C}} \mathcal{M}) \otimes_{L^1(G, C_0(X) \otimes A)} ((C_0(X) \otimes A) \rtimes G)] = \\
 &= [P \otimes_{\mathbf{C}} \mathcal{M} \otimes_{L^1(G, C_0(X) \otimes_{\max} A)} \{((C_0(X) \rtimes G) \otimes_{\max} A)\}].
 \end{aligned}$$

Let us define the following mapping of the obtained modules

$$r : p \otimes m \otimes (f \otimes a) \mapsto (p \otimes f) \otimes ma.$$

To prove that it is well defined, we have to verify the following equality

$$r((p \otimes m)h \otimes (f \otimes a)) = r((p \otimes m) \otimes h(f \otimes a)),$$

where $h \in L^1(G, C_0(X)) \otimes A$. Without loss of generality it can be assumed that

$$h(g) = \sum_{i=1}^k h^i(g) \otimes a^i,$$

where $h^i \in L^1(G, C_0(X))$, $a_i \in A$. Then

$$r((p \otimes m)h \otimes (f \otimes a)) = \sum_{i=1}^k \int_G g^{-1}(p \cdot h^i(g)) dg \otimes f \otimes m \cdot a^i a.$$

On the other hand,

$$r((p \otimes m) \otimes h(f \otimes a)) = \sum_{i=1}^k p \otimes h^i f \otimes m \cdot a^i a = \sum_{i=1}^k \int_G g^{-1}(p \cdot h^i(g)) dg \otimes f \otimes m \cdot a^i a.$$

Hence, the mapping is well defined for the algebraic tensor products. Since A is a unital algebra, r is an isomorphism of the algebraic tensor products. Consider

$$t : (P \underset{L^1(G, C_0(X))}{\otimes} (C_0(X) \rtimes G)) \underset{\mathbf{C}}{\otimes} \mathcal{M} \longrightarrow (P \underset{\mathbf{C}}{\otimes} \mathcal{M}) \underset{L^1(G, C_0(X) \otimes A)}{\otimes} ((C_0(X) \rtimes G) \otimes A),$$

$$t(p \otimes f \otimes m) := (p \otimes m) \otimes (f \otimes 1_A).$$

Then

$$\begin{aligned} t \circ r(p \otimes m \otimes (f \otimes a)) &= (p \otimes ma) \otimes (f \otimes 1_A) = \left(\frac{1}{l} p \otimes m \right) (l \otimes a) \underset{L^1}{\otimes} (f \otimes 1_A) = \\ &= \int_G g^{-1} \left(\frac{1}{l} p \otimes m \right) (l \otimes a)(g) dg \underset{L^1}{\otimes} (f \otimes 1_A) = \left(\frac{1}{l} p \otimes m \right) \otimes (lf \otimes a) = (p \otimes m) \otimes (f \otimes a), \end{aligned}$$

i. e., $t \circ r = \text{Id}$. We obtained an algebraic isomorphism of modules over $(C_0(X) \rtimes G) \otimes A$. Let us consider its behavior with the respect to norms. Considering a dense subset, we can assume that the element has the form

$$x = \sum_{i=1}^k (p_i \otimes m_i) \otimes f_i \otimes a_i,$$

where

$$f_i \otimes a_i \in L^1(G, C_0(X)) \otimes_{\max} A \subset (C_0(X) \rtimes G) \otimes_{\max} A.$$

On this dense subset r is the following isometry:

$$P \underset{\mathbf{C}}{\otimes} \mathcal{M} \otimes \{1\} \longrightarrow (P \otimes \{1\}) \underset{\mathbf{C}}{\otimes} \mathcal{M}.$$

Hence, r is an isomorphism of $(C_0(X) \rtimes G) \otimes A$ -modules and

$$\sigma\gamma\alpha = \alpha(\gamma \otimes 1).$$

Let us verify the commutativity of 10. By Lemma 5.8 it is sufficient to obtain the equality $\gamma\iota_* = \iota_*\gamma$. We have:

$$\begin{aligned} \gamma\iota_*P &= (P \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C) \underset{L^1(G, C_0(X) \otimes C)}{\otimes} ((C_0(X) \otimes C) \rtimes G) = \\ &= (P \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C) \underset{L^1(G, C_0(X)) \otimes_{\max} C}{\otimes} ((C_0(X) \rtimes G) \otimes C) = \\ &= (P \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C) \underset{L^1(G, C_0(X))}{\otimes} (C_0(X) \rtimes G), \end{aligned}$$

and

$$\begin{aligned} \iota_*\gamma P &= \left(P \underset{L^1(G, C_0(X) \otimes B)}{\otimes} (C_0(X) \otimes B) \rtimes G \right) \underset{C_0(X) \otimes B}{\otimes} C_0(X) \otimes C = \\ &= \left(P \underset{L^1(G, C_0(X)) \otimes B}{\otimes} (C_0(X) \rtimes G) \otimes B \right) \underset{B}{\otimes} C = \\ &= \left(P \underset{L^1(G, C_0(X))}{\otimes} (C_0(X) \rtimes G) \right) \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C, \end{aligned}$$

where as before $B := A \otimes \mathcal{K} \otimes C_0(\mathbf{R})$. Let us define

$$\bar{r} : \gamma\iota_*P \longrightarrow \iota_*\gamma P, \quad \bar{r}(p \otimes c \otimes h) = p \otimes h \otimes c.$$

Then \bar{r} is well defined. Indeed, if $\alpha \in A \otimes \mathcal{K} \otimes C_0(\mathbf{R})$ then

$$\bar{r}(p\alpha \otimes c \otimes h) = p\alpha \otimes h \otimes c,$$

$$\bar{r}(p \otimes \alpha c \otimes h) = p \otimes h \otimes \alpha c = p\alpha \otimes h \otimes c,$$

and if $f \in L^1(G, C_0(X))$ then

$$\bar{r}((p \otimes c)f \otimes h) = \bar{r}(pf \otimes c \otimes h) = pf \otimes h \otimes c,$$

$$\bar{r}(p \otimes c \otimes fh) = p \otimes fh \otimes c = pf \otimes h \otimes c.$$

Let $f \otimes d \in (C_0(X) \rtimes G) \otimes C$. Then

$$\bar{r}((p \otimes c \otimes h)(f \otimes d)) = \bar{r}(p \otimes cd \otimes hf) = p \otimes hf \otimes cd,$$

$$(\bar{r}(p \otimes c \otimes h))(f \otimes d) = (p \otimes c \otimes h)(f \otimes d) = p \otimes hf \otimes cd.$$

Hence, \bar{r} defines on the algebraic tensor product a well-defined homomorphism of modules over $(C_0(X) \rtimes G) \otimes C$. On the dense subset

$$(P \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C) \underset{L^1(G, C_0(X))}{\otimes} L^1(G, C_0(X)) \subset (P \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C) \underset{L^1(G, C_0(X))}{\otimes} C_0(X) \rtimes G$$

the homomorphism \bar{r} is the following isometry

$$(P \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C) \otimes \{1\} \rightarrow (P \otimes \{1\}) \underset{A \otimes \mathcal{K} \otimes C_0(\mathbf{R})}{\otimes} C.$$

Hence, \bar{r} is an isomorphism of modules over the algebra $(C_0(X) \rtimes G) \otimes C$ completed with respect to \otimes_{\max} ; and $\gamma\iota_* = \iota_*\gamma$.

Now we apply Five-lemma to the diagram (4). The proof of Theorem is completed. \square

Theorem 5.10 *If X is a separable metrizable trivial G -space, then φ_A is an isomorphism.*

Proof: Let us consider the diagram:

$$\begin{aligned}
\mathbf{K}_G^*(X; A) &\cong \mathbf{K}_G^*(C(X) \otimes A) \xrightarrow{\gamma} K_*((C(X) \otimes A) \rtimes G) \cong \\
&\downarrow \varphi_A \\
\mathbf{K}_G^*(V; A) &\cong \mathbf{K}_G^*(C(V) \otimes A) \xrightarrow{\gamma} K_*((C_0(V) \otimes A) \rtimes G) \cong \\
&\cong K_*(C(X) \otimes (A \rtimes G)) \cong K^*(X; A \rtimes G) \\
&\cong K_*(C_0(V) \otimes (A \rtimes G)) \cong K^*(V; A \rtimes G).
\end{aligned}$$

To prove the commutativity of this diagram (which implies Theorem) it is sufficient to demonstrate the following. Let \mathcal{M} be a projective $GG, C_0(V) \otimes A$ -module of finite type and E is a projective $GG, C(V)$ -module, i. e. $C(V)$ -module, because the action of G on $C(V)$ is trivial. Then

$$\gamma(\mathcal{M} \otimes_{C(V)} E) = \gamma(\mathcal{M}) \otimes_{C(V)} E.$$

Since these are “the same modules” by the definition of γ in [21] (where it was denoted by Ψ), it is necessary to verify first, that the module structure survives, and second, that the constructions behaves properly with the respect to morphisms $f : E \rightarrow F$. Indeed,

$$\begin{aligned}
(m \otimes e) \cdot_{\gamma(\mathcal{M} \otimes_{C(V)} E)} \left(\int_G a(g) U_g dg \otimes \varphi \right) &= \left(\int_G g^{-1} [(m \otimes e) \cdot (a(g) \otimes \varphi)] dg \right) = \\
&= \left(\int_G g^{-1} [(m \cdot a(g)) \otimes (e \cdot \varphi)] dg \right) = \\
&= \left(\int_G g^{-1} (m \cdot a(g)) dg \right) \otimes (e \cdot \varphi) = (m \otimes e) \cdot_{\gamma(\mathcal{M}) \otimes_{C(V)} E} \left(\int_G a(g) U_g dg \otimes \varphi \right).
\end{aligned}$$

The statement about morphisms is immediate consequence of the triviality of the action of G on $C(V)$. \square

Now, after the particular cases, we are able to prove the general theorem about the Thom isomorphism.

Theorem 5.11 *Let X be a manifold, then φ_A is an isomorphism.*

Proof: First, let us prove the theorem for the trivial bundle $X \times V$, where V is a complex finite-dimensional G -space.

Let us denote by 1 the trivial 1-dimensional G -module, and the projective space $P(V \oplus 1)$ is a compactification of V and we have the following natural homomorphism

$$j : \mathbf{K}_G(V; A) \rightarrow \mathbf{K}_G(P(V \oplus 1); A), \quad j : \mathbf{K}_G(V \times X; A) \rightarrow \mathbf{K}_G(P(V \oplus 1) \times X; A).$$

Let X be compact. Let us consider an arbitrary element $x \in \mathbf{K}_G(P(V \oplus 1) \times X; A)$. Let us consider the analytical index of the correspondent family of Dolbeault operators over $P(V \oplus 1)$ with coefficients in x (cf. [2]), i. e. an operator over C^* -algebra $C(X) \otimes A$ (see Section 9). This is an element of $\mathbf{K}_G(X; A)$. Taking the composition with j (cf. [2, p. 123]) we get a family of mappings $\alpha = \alpha_{X,A} : \mathbf{K}_G(V \times X; A) \rightarrow \mathbf{K}_G(X; A)$, having the following properties:

(a1) α is functorial with the respect to G -morphisms of X and A ;

(a2) the following diagram commutes:

$$\begin{array}{ccc} \mathbf{K}_G(V \times X; B) \otimes \mathbf{K}_G(X; A) & \longrightarrow & \mathbf{K}_G(V \times X; B \otimes A) \\ \downarrow \alpha_{X, B \otimes 1} & & \downarrow \alpha_{X, B \otimes A} \\ \mathbf{K}_G(X; B) \otimes \mathbf{K}_G(X; A) & \longrightarrow & \mathbf{K}_G(X; B \otimes A), \end{array}$$

in particular, α is a morphism of $K_G(X)$ -modules;

(a3) $\alpha_{pt, \mathbf{C}}(\lambda_V^*) = 1 \in R(G)$.

In fact, transferring the space of parameters into coefficients:

$$\mathbf{K}_G(P(V \oplus 1) \times X; B) \cong \mathbf{K}_G(P(V \oplus 1); C(X) \otimes B),$$

we reduce the situation to the case $X = pt$, and (a2) takes form

$$\begin{array}{ccc} \mathbf{K}_G(V; B) \otimes \mathbf{K}_G(pt; D) & \longrightarrow & \mathbf{K}_G(V; B \otimes D) \\ \downarrow \alpha_{pt, B \otimes 1} & & \downarrow \alpha_{pt, B \otimes D} \\ \mathbf{K}_G(pt; B) \otimes \mathbf{K}_G(pt; D) & \longrightarrow & \mathbf{K}_G(pt; B \otimes D), \end{array} \quad (5)$$

and the functoriality of (a1) means functoriality with the respect to the algebra of coefficients. In a usual way with the help of commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \mathbf{K}_G(V; A) & \longrightarrow & \mathbf{K}_G(V; A^+) & \longrightarrow & \mathbf{K}_G(V; \mathbf{C}) \\ & & \downarrow \alpha & & \downarrow \alpha \\ 0 \rightarrow \mathbf{K}_G(pt; A) & \longrightarrow & \mathbf{K}_G(pt; A^+) & \longrightarrow & \mathbf{K}_G(pt; \mathbf{C}) \end{array}$$

α can be extended to non-unital (in particular, non-compact) case. The diagram (5) is still commutative. We will need the following two particular cases of it. For $B = C_0(X) \otimes A$, $D = C_0(V)$ we get for $x \in \mathbf{K}_G(V \times X; A)$, $y \in K_G(V)$

$$\alpha_{X \times V, A}(x \otimes y) = \alpha_{X, A}(x) \otimes y \in \mathbf{K}_G(X \times V; A). \quad (6)$$

For $D = C_0(X) \otimes A$, $B = C_0(W)$ we get for $y' \in \mathbf{K}_G(W \times X; A)$, $x' \in K_G(V)$

$$\alpha_{W \times X, A}(x' \otimes y') = \alpha_{pt, \mathbf{C}}(x') \otimes y' \in \mathbf{K}_G(W \times X; A). \quad (7)$$

Let $x \in \mathbf{K}_G(X; A)$, then by (7) with $W = 0$ and (a2)

$$\alpha(\lambda_V^* x) = \alpha(\lambda_V^*) x = x. \quad (8)$$

Let $y \in \mathbf{K}_G(V \times X; A) = \mathbf{K}_G(V; C_0(X) \otimes A)$, then by (6) and (7) with $W = V$

$$\alpha(y) \otimes \lambda_V^* = \alpha(y \otimes \lambda_V^*) = \alpha(\lambda_V^* \otimes \tilde{y}) = \alpha(\lambda_V^*) \tilde{y} = \tilde{y} \in \mathbf{K}_G(X \times V; A), \quad (9)$$

where by $\tilde{y} \in \mathbf{K}_G(X \times V; A)$ we denote the element, obtained from y under the mapping $X \times V \rightarrow V \times X$, $(x, v) \mapsto (-v, x)$ (such that $V \times X \times V \rightarrow V \times X \times V$, $(u, x, v) \mapsto (-v, x, u)$ is homotopic to the identity). Let us apply to the both parts of (9) the isomorphism $\mathbf{K}_G(X \times V; A) \rightarrow \mathbf{K}_G(V \times X; A)$. We obtain that $\varphi\alpha$ is an isomorphism. But by (8) $\alpha\varphi = \text{Id}$, hence, α is the two-sides inverse. And the unknown automorphism is the identity.

The pass to the case of general complex H -bundle $E \rightarrow Y$ we make similarly to [2, p. 124]. Namely, let us take $G = U(n) \times H$ and the principal G -bundle X , with which E is associated, $Y = X/U(n)$, in such a way that

$$\mathbf{K}_G(X; A) \cong \mathbf{K}_H(Y; A), \quad \mathbf{K}_G(V \times X; A) \cong \mathbf{K}_H((V \times X)/U(n); A) \cong \mathbf{K}_H(E; A).$$

Since the identifications commute with multiplication by λ_V^* , we get the desired result. (Other way is to enclose the bundle into a trivial one and use the transitivity of the Thom homomorphism). \square

We shall pass now to some further constructions, connected with the Thom homomorphism. They will be necessary for the definition of topological index.

Let X and Y be smooth G -manifolds, $i : X \rightarrow Y$ the equivariant enclosure, Y is equipped with a G -invariant Riemannian metric, (TX, p_T, X) is the tangent bundle of X , (N, p_N, X) is the normal bundle for i . Let us choose a function $\varepsilon : X \rightarrow (0, \infty)$ such that the map of N to itself

$$n \mapsto \varepsilon \frac{n}{1 + |n|}$$

is G -equivariant and determines a G -diffeomorphism $\Phi : N \rightarrow W$ on an open tubular neighborhood $W \supset X$ in Y . The enclosure $i : X \rightarrow Y$ is decomposed in a composition of two enclosures $i_1 : X \rightarrow W$ and $i_2 : W \rightarrow Y$. Passing to differentials we obtain

$$TX \xrightarrow{di_1} TW \xrightarrow{di_2} TY, \quad d\Phi : TN \rightarrow TW.$$

Lemma 5.12 [16, p. 112] *The manifold TN can be identified with $p_T^*(N \oplus N)$ with the help of a G -diffeomorphism ψ such that the following diagram is commutative*

$$\begin{array}{ccc} p_T^*(N \oplus N) & \xleftarrow{\psi} & TN \\ \downarrow & & \downarrow \\ TX & & N \\ & \searrow p_T \quad \swarrow p_N & \\ & X & \end{array}$$

Proof: The manifold TN as the vector bundle over N can be identified with $p_N^*(TX) \oplus p_N^*(N)$. A point of the total space TN is a pair of the form $(n_1, t + n_2)$, where both vectors are from the fiber over the point $x \in X$. Similarly, we represent elements $p_T^*(N \oplus N)$ as pairs of the form $(t, n_1 + n_2)$. Let us define ψ by the equality $\psi(n_1, t + n_2) = (t, n_1 + n_2)$. \square

With the help of the relation $i \cdot (n_1, n_2) = (-n_2, n_1)$, we can equip

$$p_T^*(N \oplus N) = p_T^*(N) \oplus p_T^*(N)$$

with a structure of a complex manifold. Then we can consider the Thom homomorphism

$$\varphi : \mathbf{K}_G(TX; A) \rightarrow \mathbf{K}_G(p_T^*(N \oplus N), A).$$

Since TW is an open G -stable subset of TY and $di_2 : TW \rightarrow TY$ is an enclosure, by the construction 3.20, there is the homomorphism $(di_2)_* : K_G(TW; A) \rightarrow K_G(TY; A)$.

Definition 5.13 Let $i : X \rightarrow Y$ be an enclosure. The *Gysin homomorphism* is the mapping

$$i_! : \mathbf{K}_G(TX; A) \rightarrow \mathbf{K}_G(TY; A), \quad i_! = (di_2)_*(d\Phi^{-1})^*\psi^*\varphi.$$

In other words, it is obtained by passage to K -groups in the upper part of the diagram

$$\begin{array}{ccccccc} p_T^*(N \oplus N) & \xleftarrow{\psi} & TN & \xrightarrow{d\Phi} & TW & \xrightarrow{di_2} & TY \\ q_T \downarrow & & \downarrow & & \downarrow & & \downarrow \\ TX & & N & & W & & Y \\ & \searrow p_T & \swarrow p_N & \searrow \Phi & \downarrow i_2 & & \\ & & X & \xrightarrow{i_1} & W & \xrightarrow{i_2} & Y \end{array}$$

Another choice of metric and neighborhood W induces the homotopic map and (by the item 3 of Theorem 5.14 below) the same homomorphism.

Theorem 5.14 () 1. $i_!$ is a homomorphism of $R(G)$ -modules.

2. Let $i : X \rightarrow Y$ and $j : Y \rightarrow Z$ be two G -inclusions, then $(j \circ i)_! = j_! \circ i_!$.

3. Let enclosures $i_1 : X \rightarrow Y$ and $i_2 : X \rightarrow Y$ are G -homotopic in the class of enclosures. Then $(i_1)_! = (i_2)_!$.

4. Let $i_! : X \rightarrow Y$ be a G -diffeomorphism, then $i_! = (di^{-1})^*$.

5. An enclosure $i : X \rightarrow Y$ can be represented as a compositions of enclosures X in N (as the zero section $s_0 : x \rightarrow N$) and $N \rightarrow Y$ by $i_2 \circ \Phi : N \rightarrow Y$. Then $i_! = (i_2 \circ \Phi)_!(s_0)_!$.

6. Consider the complex bundle $p_T^*(N \otimes \mathbf{C})$ over TX . Let us form the complex $\Lambda(p_T^*(N \otimes \mathbf{C}), 0)$:

$$0 \rightarrow \Lambda^0(p_T^*(N \otimes \mathbf{C})) \xrightarrow{0} \dots \xrightarrow{0} \Lambda^k(p_T^*(N \otimes \mathbf{C})) \rightarrow 0$$

with the noncompact support. If $a \in \mathbf{K}_G(TX; A)$, then the complex $a \otimes \Lambda(p_T^*(N \otimes \mathbf{C}), 0)$ has the compact support and determines an element of $\mathbf{K}_G(TX; A)$. Then $(di)^*i_!(a) = a \cdot \Lambda(p_T^*(N \otimes \mathbf{C}), 0)$.

7. For $x \in \mathbf{K}_G(TX; A)$ and $y \in K_G(TY)$ $i_!(x(di)^*y) = i_!(x) \cdot y$.

Proof: 1. By the definition of $i_!$.

2. To simplify the argument, let us identify the tubular neighborhood with the normal bundle. Then $(j \circ i)_!$ is the composition

$$\mathbf{K}_G(TX; A) \xrightarrow{\varphi} \mathbf{K}_G(TN \oplus TN'_1; A) \rightarrow \mathbf{K}_G(TZ; A),$$

where N' is the normal bundle of Y in Z , $N'_1 = N'|_X$, and $TN \oplus TN'_1$ is considered in the same way as at p. 29 as a complex bundle over TX . On the other hand, $j_! \circ i_!$ represents the composition

$$\begin{aligned} \mathbf{K}_G(TX; A) &\xrightarrow{\varphi} \mathbf{K}_G(TN; A) \rightarrow \mathbf{K}_G(TY; A) \xrightarrow{\varphi} \\ &\xrightarrow{\varphi} \mathbf{K}_G(TN'; A) \rightarrow \mathbf{K}_G(TZ; A). \end{aligned}$$

By properties of φ , the following diagram is commutative

$$\begin{array}{ccccc} \mathbf{K}_G(TX; A) & \xrightarrow{\varphi} & \mathbf{K}_G(TN; A) & \rightarrow & \mathbf{K}_G(TY; A) \\ & \searrow \varphi & \downarrow \varphi & & \downarrow \varphi \\ & & \mathbf{K}_G(TN \oplus TN'_1; A) & \xrightarrow{\varphi} & \mathbf{K}_G(TN'; A) \\ & & \downarrow & & \downarrow \\ & & \mathbf{K}_G(TZ; A) & = & \mathbf{K}_G(TZ; A), \end{array}$$

This completes the proof of item 2.

3. The homotopy of enclosures does not influence on q_T , but only on the further maps in the definition of the Gysin homomorphism. But for these maps the assertion follows from the homotopy invariance of K -theory.

4. In this case $N = X$, $W = Y$, $\Phi = i$, $i_2 = \text{Id}_Y$, and the formula is obvious.

5. By 2.

6. By definition,

$$(di)^*i_! = (di_1)^*(di_2)^*(di_2)_*(d\varphi^{-1})^*\psi^*\varphi^* = (\psi \circ d\Phi^{-1} \circ di_1)^* \circ \varphi,$$

where $i_1 : X \rightarrow W$, $i_2 : W \rightarrow Y$. Let $(n_1, t + n_2) \in TN = p_N^*(TX) \oplus p_N^*(N)$, where n_1 is the shift under the exponential mapping, $t + n_2$ a tangent vector to W . If $d\Phi(n_1, t + n_2)$ is in TX , then $n_1 = n_2 = 0$. Hence,

$$d\Phi^{-1}di_1(t) = (0, t + 0), \quad \psi \circ d\Phi^{-1} \circ di_1(t) = (t, 0 + 0).$$

Therefore, $\psi \circ d\Phi^{-1} \circ di_1 : TX \rightarrow p_T^*(N \oplus N)$ is the enclosure of the zero section. Since, $\varphi(a) = a \cdot \Lambda(q_T^*p_T^*(N \otimes \mathbf{C}), s_{p_T^*(N \otimes \mathbf{C})})$, it follows that $(di)^*i_!(a) = a \cdot \Lambda(p_T^*(N \otimes \mathbf{C}), 0)$.

7. The mapping $di_1 \circ q_T \circ \psi \circ d\Phi^{-1} : TW \rightarrow TW$ is homotopic to the identical mapping. Hence,

$$\begin{aligned} i_!(x \cdot (di)^*y) &= (di_2)_*(d\Phi^{-1})^*\psi^*\varphi(x \cdot (di)^*y) = \\ &= (di_2)_*(d\Phi^{-1})^*\psi^*[(q_T^*(x)\lambda_{p_T^*(N \otimes \mathbf{C})})(q_T^*(di)^*y)] = \\ &= (di_2)_*[(d\Phi^{-1})^*\psi^*(q_T^*(x)\lambda_{p_T^*(N \otimes \mathbf{C})}) \underbrace{(d\Phi^{-1})^*\psi^*q_T^*(di_1)^*(di_2)^*y}_{\text{Id}}] = \\ &= [(di_2)_*(d\Phi^{-1})^*\psi^*(q_T^*(x)\lambda_{p_T^*(N \otimes \mathbf{C})})] [(di_2)_*(di_2)^*y] = i_!(x) \cdot y. \quad \square \end{aligned}$$

Theorem 5.15 1. *Let V be a G - \mathbf{R} -vector space and $X = pt$ a trivial G -manifold. Hence, $TX = pt$ and $TV = V \otimes \mathbf{C}$. Consider the enclosure of zero $i : X \rightarrow V$. Then the mapping*

$$i_! : \mathbf{K}^G(A) = \mathbf{K}_G(TX; A) \rightarrow \mathbf{K}_G(TV; A) = \mathbf{K}_G(V \otimes \mathbf{C}; A)$$

coincides with the Thom homomorphism $\varphi_A^{V \otimes \mathbf{C}}$.

2. Let V_1 and V_2 be G - \mathbf{R} -spaces, $i : X \rightarrow V_1$ an enclosure. Let us define the enclosure $k : X \rightarrow V_1 \oplus V_2$ by the formula $k(x) = i(x) + 0$. Then the following diagram is commutative

$$\begin{array}{ccc} & & \mathbf{K}_G(TV_1; A) \\ & \nearrow i_! & \downarrow \varphi \\ \mathbf{K}_G(TX; A) & & \\ & \searrow k_! & \downarrow \\ & & \mathbf{K}_G(T(V_1 \oplus V_2), A), \end{array}$$

where φ the Thom homomorphism of the complex bundle

$$T(V_1 \oplus V_2) = V_1 \otimes \mathbf{C} \oplus V_2 \otimes \mathbf{C} \longrightarrow TV_1 = V_1 \otimes \mathbf{C}.$$

Proof: 1. The assertion follows from the definition of $i_!$. More precisely, $X = 0 \in V$, $N = V$. Also, W can be chosen equal to the interior D_1 of the ball of radius 1 in V with respect to an invariant metric. In this case the diagram from the definition of the Gysin homomorphism 5.13 takes the following form

$$\begin{array}{ccccccc} V \otimes \mathbf{C} & \xrightarrow{\Psi} & V \otimes \mathbf{C} & \xrightarrow{d\Phi} & D_1 \otimes \mathbf{C} & \xrightarrow{di_2} & V \otimes \mathbf{C} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ TX = 0 & & V & & D_1 & & V \\ & \searrow & \swarrow \Phi & & \downarrow i_2 & & \\ & & X = 0 & \xrightarrow{i_1} & D_1 & \xrightarrow{i_2} & V. \end{array}$$

In our case $\Psi = \text{Id}$ and $di_2 \circ d\Phi$ is homotopic to Id , since it has the form $v \otimes z \mapsto (v \otimes z)/(1 + |v \otimes z|)$. Hence, $i_! = \varphi$.

2. Let the enclosure $i : X \rightarrow V_1$ has the normal bundle N and the tubular neighborhood W . Then $N \oplus V_2$ is a normal bundle for the enclosure k with the tubular neighborhood $W \oplus D(V_2)$, where $D(V_2)$ is a ball. If $a \in \mathbf{K}_G(TX; A)$, then

$$k_!(a) = (di_2 \oplus 1)_*(d\Phi^{-1} \oplus 1)^*(\psi \oplus 1)^*\varphi_A^{N \oplus N \oplus V_2 \oplus V_2}(a).$$

By item 2 of theorem 5.5, $\varphi_A^{N \oplus N \oplus V_2 \oplus V_2} = \varphi_A^{N \oplus N} \cdot \varphi_{\mathbf{C}}^{V_2 \oplus V_2}$. Since $a = a \cdot \underline{\mathbf{C}}$, where $\underline{\mathbf{C}}$ is the trivial line bundle,

$$\begin{aligned} k_!(a) &= (di_2)_*(d\Phi^{-1})^*\Psi^*\varphi_A^{N \oplus N}(a) \cdot \varphi_{\mathbf{C}}^{V_2 \oplus V_2}(\underline{\mathbf{C}}) = \\ &= i_!(a) \cdot \lambda_{T(V_1 \oplus V_2)} = \varphi(i_!(a)). \quad \square \end{aligned}$$

6 Analytical index

let us remind how a pseudo differential operator $\chi(\sigma)$ over A can be defined starting from a symbol σ in the case $G = e$ (see, e.g. [39]).

Suppose X is a compact closed smooth manifold, $\pi : T^*X \rightarrow X$ is the projection of the cotangent bundle, E_1 and E_2 are smooth A -bundles over X . Let $\{U_j\}$ be a trivializing cover of X for E_1 and E_2 , $\{\varphi_j\}$ be the subordinate partition of unity, $\{\psi_j\}$ a collection of functions such that $\psi_j|_{\text{supp } \varphi_j} \equiv 1$. A symbol of order m is a morphism of A -bundles $\sigma : \pi^*E_1 \rightarrow \pi^*E_2$.

Let $u_j \in \Gamma_0^\infty(U_j, E_1)$ be a smooth section tending to 0 at infinity. Let us assume

$$(\chi_j \sigma)(u_j)(x) := \frac{1}{(2\pi)^n} \int e^{i\langle x, \xi \rangle} \sigma(x, \xi) \int e^{-i\langle y, \xi \rangle} u_j(y) dy d\xi.$$

For $u \in \Gamma^\infty(X, E_1)$ let us define $\chi\sigma$ by the formula

$$(\chi\sigma)(u)(x) = \sum_j \psi_j(x) (\chi_j \sigma)(\varphi_j u)(x).$$

This is the section of the symbol-map (see, e. g. [39, §2.1]). We obtain a bounded A -homomorphism admitting an adjoint in Sobolev spaces:

$$\chi\sigma : H^s(E_1) \cong l_2(\mathcal{P}_1) \rightarrow l_2(\mathcal{P}_2) \cong H^{s-m}(E_2).$$

Suppose G is an arbitrary compact Lie group, X is a compact G -manifold, E_1 and E_2 are GGA -bundles over X , the symbol $\sigma : \pi^*E_1 \rightarrow \pi^*E_2$ is a GGA -morphism. The action of G on Sobolev spaces is naturally defined by the formula $g(u)(x) = g(u(g^{-1}x))$. Without loss of generality, we can assume the Sobolev inner products on $H^s(E_i) \cong l_2(\mathcal{P}_i)$ to be GGA -products (see Remark 2.8).

More precisely, let us choose a C^* -Hermitian metrics in bundles to be G -invariant, we obtain that under the action of G on Sobolev spaces we get each time admissible products (see, e. g. [39, §2.1]). Hence, this Ψ DO admits an adjoint with the respect to the averaged Sobolev product.

Suppose σ is an elliptic symbol, then (see, e. g. [39, §2.1])

$$(\chi\sigma)(\chi\tau) = 1 + K_1, \quad (\chi\tau)(\chi\sigma) = 1 + K_2 \quad (10)$$

for some symbol (parametrix) τ , where K_1 and K_2 are A -compact operators. The action of group G on Ψ DO is given by $g(P)u = g(P(g^{-1}(u)))$.

Let $g \in G$ be fixed. Let us denote by $g\{\}$ the action of g on sections and by $g\cdot$ the action on the elements of total space of a bundle. By [39, §2.1] OP_{r-1} and CZ_r , hence, Int_r are invariant under diffeomorphisms, in particular, under g . Hence Int_r has this invariance property. Moreover (see, e. g. [39, §2.1]),

$$\begin{aligned} \sigma_m(g(\chi\sigma))(x, \xi) &= \sigma_m(g \cdot \chi\sigma \cdot g)(x, \xi) = \\ &= \sigma_m(\chi\sigma)(gx, g\xi) = \sigma(gx, g\xi) = \sigma(x, \xi), \end{aligned}$$

since g acts on covectors by

$$g\xi = g_*\xi = (D(g^{-1}))^*\xi,$$

where the tangential mapping (derivative) of an arbitrary smooth map φ we denote by $D(\varphi)$. Hence

$$g(\chi\sigma) = g\{\chi\sigma\}g^{-1} = \chi'\sigma,$$

i. e., $g(\chi\sigma)$ is another Ψ DO with the same symbol σ , where χ' is some other section of the symbol-map.

Lemma 6.1 *Let us fix $D = \chi\sigma$. Then the map $G \rightarrow \text{Int}_m(E, F)$, given by the formula*

$$g \mapsto g(D), \quad g(D)(u) = g\{Dg^{-1}\{u\}\},$$

is continuous, i. e., for any s this map is continuous as a map from the group G to the space of operators $H^s \rightarrow H^{s-m}$ equipped with the uniform topology.

Proof: (cf. [6]) Let $a \in \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of group G . The action of a defines a vector field \vec{a}_x . The corresponding differential operator a_E acts on sections of the bundle E :

$$a_E(u) = \left\langle \vec{a}_x, \frac{\vec{\partial}}{\partial x_i} \right\rangle (u).$$

The symbol of this operator is equal to $\sigma_{a,E} = \langle \vec{a}, \xi \rangle \text{Id}_E$. Similarly for F . Then $\sigma\sigma_{a,E} = \sigma_{a,F}\sigma$ (see, e. g. [39, §2.1]),

$$D(\chi\sigma_{a,E}) - (\chi\sigma_{a,F})D \in \text{Int}_m(E, F),$$

$$\|D(\chi\sigma_{a,E}) - (\chi\sigma_{a,F})D\|_s^m \leq \|D\|(\|a_E\| + \|a_F\|),$$

where $a_E = \chi\sigma_{a,E}$. If a is in a bounded neighborhood of $0 \in \mathfrak{g}$, there exist numbers c_s such that

$$\|Da_E - a_FD\|_s^m < c_s,$$

where $\|\cdot\|_s^m$ is the norm in Int_m , i. e., the uniform norm in the space of operators $H^s \rightarrow H^{s-m}$.

For $g_t = \exp ta$ and a smooth section u let us assume $f(t) := g_t(D)u$. Then

$$f_t(t) = \exp(ta_F)D\exp(-ta_E)u.$$

Indeed, it is sufficient to prove this equality in one chart for analytical functions, because the analytical basis was constructed in an explicit form in [31, p. 854]. For an analytical section $u(x)$ we have to prove that

$$(e^{ta_E}u)(x) = u(\varphi_x(t)), \tag{11}$$

where $\varphi_x(t)$ is the orbit of the action of the one-parameter subgroup, generated by a . For this purpose let us consider the function $F(t) = u(\varphi_x(t))$. It is analytical for $|t| \leq 1$. Then for $|t| \leq 1$ the series

$$F(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} t^n$$

is convergent. Since $\varphi_x(t)$ is an integral curve of the vector field \vec{a}_x ,

$$F'(t) = \frac{du(\varphi_x(t))}{dt} = \frac{d\varphi_x(t)}{dt}u = \vec{a}_{\varphi_x(t)}u = (a_E u)(\varphi_x(t)),$$

i. e., $F'(t)$ plays the role of $F(t)$ for the section $a_E u$. By induction

$$F^{(n)}(t) = (a_E^n u)(\varphi_x(t)).$$

Therefore

$$F^{(n)}(0) = (a_E^n u)(\varphi_x(0)) = (a_E^n u)(x).$$

Hence

$$F(t) = \sum_{n=0}^{\infty} \frac{(a_E^n u)(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{(ta_E)^n u}{n!}(x) = (e^{ta_E} u)(x),$$

and we have (11). Further, since the action of G on H^s is an isometry,

$$\left\| \frac{df_t}{dt} \right\|_{m-s} = \left\| \exp ta_F (D a_E - a_F D) \exp(-ta_E) u \right\| \leq c_s \|u\|_s.$$

This implies continuity. \square

Let $\chi^1 \sigma = D_1$ and $\chi^2 \sigma = D_2$ be two Ψ DO with the same symbol σ . Then $D_1 - D_2 \in \mathcal{K}(l_2(\mathcal{P}_1, \mathcal{P}_2))$. If σ is a G -equivariant symbol, then we have shown that $g(D) = D' = \chi' \sigma$. Hence the orbit $G(D)$ is in the closed linear manifold

$$D + \mathcal{K}(l_2(\mathcal{P}_1), l_2(\mathcal{P}_2)).$$

By the previous lemma there exists the averaged operator $\text{Av } D$ and it lies in this linear manifold. Thus for an elliptic G -symbol the operator $\text{Av } D$ is an A -Fredholm one. Really, from (10) it follows

$$\begin{aligned} \text{Av } (D) \text{Av } (\chi \tau) &= (\chi \sigma + K_1')(\chi \tau + K_1'') = \\ &= (\chi \sigma)(\chi \tau) + K_1'(\chi \tau) + (\chi \sigma) K_1'' + K_1' K_1'' = 1 + \widetilde{K}_1, \\ \text{Av } (\chi \tau) \text{Av } (D) &= (\chi \tau + K_1'')(\chi \sigma + K_1') = \\ &= (\chi \tau)(\chi \sigma) + K_1''(\chi \sigma) + (\chi \tau) K_1' + K_1'' K_1' = 1 + \widetilde{K}_2, \end{aligned}$$

where K_1' , K_1'' , \widetilde{K}_1 , \widetilde{K}_2 are A -compact operators. It was proved that each Ψ DO over unital C^* -algebra admits an adjoint. Hence each element of the orbit $G(D)$ has an adjoint. Since the operators admitting an adjoint form a Banach space (algebra), the averaged operator $\text{Av } D$ has an adjoint. It remains to apply our theory of Fredholm operators. Thus the equivariant analytical index

$$\text{a-ind } \sigma = \text{index Av } D \in K^G(A) = \mathbf{K}_G^0(\text{pt}; A).$$

is defined. It is clear, that all argument about homotopies (see, e. g. [39]) is valid in the equivariant case. Hence in this case the analytical index defines a homomorphism of Abelian groups

$$\text{a-ind}_G^X : \mathbf{K}_G(TX; A) \rightarrow K^G(A) = \mathbf{K}_G^0(\text{pt}; A).$$

7 The axiomatic approach

Let us start from the definition of the topological index. Let G be a compact Lie group and X be a compact G -manifold. From [33] it follows that there exists a representation of G in orthogonal group OV of some real finite-dimensional space V and G -enclosure $i : X \rightarrow V$. Thus, the Gysin homomorphism (see 5):

$$i_! : \mathbf{K}_G(TX; A) \rightarrow \mathbf{K}_G(TV; A) = \mathbf{K}_G(V \otimes \mathbf{C}; A)$$

is defined. Since $TV = V \otimes \mathbf{C}$ is a complex vector space, we have the following Thom isomorphism (see 5):

$$\varphi : K_0^G(A) = \mathbf{K}_G^0(pt; A) \rightarrow \mathbf{K}_G(TV; A).$$

Definition 7.1 The *topological index* is the following map:

$$\text{t-ind}_G^X : \mathbf{K}_G(TX; A) \rightarrow K^G(A), \quad \text{t-ind}_G^X := \varphi^{-1} \circ i_!.$$

Theorem 7.2 1. The index t-ind_G^X does not depend on the choice of V , enclosure $i : X \rightarrow V$, and representation $G \rightarrow O(V)$.

2. The index t-ind_G^X is a $R(G)$ -homomorphism.

3. If $X = pt$, then the map

$$\text{t-ind}_G^X : K^G(A) = \mathbf{K}_G(TX; A) \rightarrow K^G(A)$$

coincides with $\text{Id}_{K^G(A)}$.

4. Suppose X and Y are compact G -manifolds, $i : X \rightarrow Y$ is a G -enclosure. Then the diagram

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) & \xrightarrow{i_!} & \mathbf{K}_G(TY; A) \\ & \searrow \text{t-ind}_G^X & \swarrow \text{t-ind}_G^Y \\ & K^G(A) & \end{array}$$

commutes.

Proof: 1). Let us consider the enclosures

$$i_1 : X \rightarrow V_1, \quad i_2 : X \rightarrow V_2.$$

Denote by $j = i_1 + i_2$ the induced enclosure $j : X \rightarrow V_1 \oplus V_2$. It is sufficient to show that the topological index, which comes from i_1 , coincides with the index, which comes from j . Let us define a homotopy of G -enclosures by the formula

$$j_s(x) = i_1(x) + s \cdot i_2(x) : X \rightarrow V_1 \oplus V_2, \quad 0 \leq s \leq 1.$$

Then by Theorems 5.14.3 and 5.15.1 the indexes for j and j_0 coincide. Let us show now that $j_0 = i_1 + 0$ and i_1 define the same topological indexes. For this purpose consider the diagram

$$\begin{array}{ccc}
 & \mathbf{K}_G(TX; A) & \\
 (i_1)_! \swarrow & & \searrow (j_0)_! \\
 \mathbf{K}_G(TV_1; A) & \xrightarrow{\varphi_2} & \mathbf{K}_G(T(V_1 \oplus V_2); A), \\
 \swarrow \varphi_1 & & \searrow \varphi_3 \\
 & K^G(A) &
 \end{array}$$

where φ_i are the corresponding Thom homomorphisms. The upper triangle is commutative by Theorem 5.15.2, and the lower is commutative by Proposition 5.6. Hence $\varphi_1^{-1} \circ (i_1)_! = \varphi_3^{-1} \circ (j_0)_!$ as desired.

2). This statement follows from 5.3 and 5.14.1.

3). This follows immediately from the definition of the index and from 5.15.1

4). Let us consider the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{i} & Y \\
 & \searrow j \circ i & \swarrow j \\
 & V. &
 \end{array}$$

Let us apply 5.14.2. We have the following commutative diagram

$$\begin{array}{ccc}
 \mathbf{K}_G(TX; A) & \xrightarrow{i_!} & \mathbf{K}_G(TY; A) \\
 & \searrow (j \circ i)_! & \swarrow j_! \\
 & \mathbf{K}_G(TV; A) &
 \end{array}$$

or

$$\begin{array}{ccc}
 \mathbf{K}_G(TX; A) & \xrightarrow{i_!} & \mathbf{K}_G(TY; A) \\
 & \searrow (j \circ i)_! & \swarrow j_! \\
 & \mathbf{K}_G(TV; A) & \\
 \text{t-ind}_G^X \swarrow & \uparrow \varphi & \searrow \text{t-ind}_G^Y \\
 & K^G(A). &
 \end{array}$$

□

Definition 7.3 An *index function* is a family of $R(G)$ -homomorphisms $\{\text{ind}_G^X\}$

$$\text{ind}_G^X : \mathbf{K}_G(TX; A) \rightarrow K^G(A),$$

where G runs through the set of compact Lie groups and X is a smooth compact G -manifold. This family is restricted to satisfy the following two conditions:

1. If $f : X \rightarrow Y$ is a G -diffeomorphism, then the diagram

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) & \xrightarrow{(df^{-1})^*} & \mathbf{K}_G(TY; A) \\ & \searrow \text{ind}_G^X \quad \swarrow \text{ind}_G^Y & \\ & K^G(A) & \end{array}$$

is commutative.

2. If $\psi : H \rightarrow G$ is a homomorphism of groups, then the diagram

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) & \xrightarrow{\psi^*} & \mathbf{K}_H(TX; A) \\ \text{ind}_G^X \downarrow & & \downarrow \text{ind}_H^X \\ \mathbf{K}^G(A) & \xrightarrow{\psi^*} & \mathbf{K}^H(A) \end{array}$$

is commutative.

Assertion 7.4 *The topological index t-ind_G^X is an index function.*

Proof: 1). Suppose $i : Y \hookrightarrow V$, $j = i \circ f : X \hookrightarrow V$. By 5.14.2, the following diagram is commutative

$$\begin{array}{ccc} & \mathbf{K}_G(TY; A) & \\ & \uparrow f_! & \\ \mathbf{K}_G(TX; A) & \xrightarrow{j_!} & \mathbf{K}_G(TV; A) \xrightarrow{\varphi^{-1}} K^G(A). \end{array}$$

By 5.14.4, we have in this case $f_! = (df^{-1})^*$ and it remains to use the definition of t-ind .

2). Immediately follows from the definition. \square

Let us consider the following two axioms.

Axiom A1. If $X = \text{onpoint}$, then $\text{ind}_G^X : \mathbf{K}_G(TX; A) \rightarrow K^G(A)$ coincides with $\text{Id}_{K^G(A)}$.

Axiom A2. Suppose $i : X \rightarrow Y$ is a G -enclosure, then the diagram

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) & \xrightarrow{i_!} & \mathbf{K}_G(TY; A) \\ & \searrow \text{ind}_G^X \quad \swarrow \text{ind}_G^Y & \\ & K^G(A) & \end{array}$$

is commutative.

Corollary 7.5 (from Theorem 7.2) *The topological index t-ind_G^X satisfies axioms A1 and A2.* \square

Theorem 7.6 *Let ind_G^X be an index function satisfying axioms A1 and A2. Then $\text{ind}_G^X = \text{t-ind}_G^X$.*

Proof: Consider a G -enclosure $i : X \rightarrow V$ of a manifold X in a real vector G -space V . The one-point compactification V^+ (i. e., a sphere) is a G -manifold with the canonical G -inclusion $\varepsilon^+ : V \rightarrow V^+$. Suppose $i^+ = \varepsilon^+ \circ i : X \rightarrow V^+$. If $P = 0 \in V$ and $j : P \rightarrow V$ is the inclusion, then we obtain the diagram

$$\begin{array}{ccccc}
 & & \mathbf{K}_G(TX; A) & & \\
 & \swarrow i_! & \downarrow i_!^+ & \searrow \text{ind}_G^X & \\
 \mathbf{K}_G(TV; A) & \xrightarrow{(\varepsilon^+)_!} & \mathbf{K}_G(TV^+; A) & \xrightarrow{\text{ind}_G^{V^+}} & K^G(A) \\
 & \nwarrow j_! & \uparrow j_!^+ & \nearrow \text{ind}_G^P & \\
 & & \mathbf{K}_G(TP; A) = K^G(A) & &
 \end{array}$$

where $j^+ = \varepsilon^+ \circ j : P \rightarrow V^+$. By 5.14.2 (resp., by axiom A2), the left (resp., right) triangles commute. By A1, ind_G^P is the identity mapping. Hence

$$\begin{aligned}
 \text{ind}_G^X &= \text{ind}_G^{V^+} i_!^+ = \text{ind}_G^{V^+} (\varepsilon^+)_! i_! = \text{ind}_G^{V^+} (j^+)_! j_!^{-1} i_! = \\
 &= \text{ind}_G^P j_!^{-1} i_! = j_!^{-1} i_! = \text{t-ind}_G^X.
 \end{aligned}$$

Since $j_! : K^G(A) \rightarrow \mathbf{K}_G(TV; A) = \mathbf{K}_G(V \otimes \mathbf{C}; A)$ coincides with the Thom homomorphism, the theorem is proved. \square

Axiom B1 (excision). Let U be a (noncompact) G -manifold and

$$j_1 : U \rightarrow X_1, \quad j_2 : U \rightarrow X_2$$

be G -enclosures of the manifold U on open subsets of compact G -manifolds X_1 and X_2 . Then the diagram

$$\begin{array}{ccccc}
 & & \mathbf{K}_G(TX_1; A) & & \\
 & \nearrow (dj_1)_* & & \searrow \text{ind}_G^{X_1} & \\
 \mathbf{K}_G(TU; A) & & & & K^G(A) \\
 & \searrow (dj_2)_* & & \nearrow \text{ind}_G^{X_2} & \\
 & & \mathbf{K}_G(TX_2; A) & &
 \end{array}$$

is commutative.

Suppose there exists though one of indicated enclosures. Then by the axiom, the index

$$\text{ind}_G^U : \mathbf{K}_G(TU; A) \rightarrow K^G(A)$$

can be well defined.

Let us denote by $\mathbf{C}\text{-ind}_H^Y$ the classical (complex) index

$$\mathbf{C}\text{-ind}_H^Y : K_H(TY) \rightarrow R(H).$$

We have the following statement (see [6]). Suppose $j : * \rightarrow \mathbf{R}^n$ is the enclosure of $\vec{0}$, hence $j_! : R(O(n)) \rightarrow K_{O(n)}(T\mathbf{R}^n)$. Then $\mathbf{C}\text{-ind}_{O(n)}^{\mathbf{R}^n} j_!(1) = 1$.

Let $\pi : P \rightarrow X$ be a compact differentiable principal bundle for a group H (compact Lie group). Then we have (right) free action of H on P and $X = P/H$. Suppose we have left action of G on P and these two actions commute. Let F be a compact left $(G \times H)$ -manifold. We can form the associated bundle $\pi_1 : Y = P \times_H F \rightarrow X$ with the natural action of G . Consider the tangent bundle along the fibers of π_1 . Let us denote it by $T_F Y$. Then $T_F Y$ is a G -invariant real subbundle of TY and $T_F Y = P \times_H TF$. Using the metric it is possible to decompose TY into a direct sum $TY = T_F Y \oplus \pi_1^*(TX)$. Therefore the multiplication

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) \otimes K_G(T_F Y) & & \\ \downarrow & & \\ \mathbf{K}_G(\pi_1^* TX; A) \otimes K_G(T_F Y) & \rightarrow & \mathbf{K}_G(TY; A) \end{array}$$

is defined. There exists the map

$$K_{G \times H}(TF) \rightarrow K_{G \times H}(P \times TF) \cong K_G(P \times_H TF) = K_G(T_F Y).$$

Hence we can define the mapping

$$\gamma : \mathbf{K}_G(TX; A) \otimes K_{G \times H}(TF) \rightarrow \mathbf{K}_G(TY; A).$$

Let us denote $\gamma(a \otimes b)$ by $a \cdot b$.

If V is a complex vector $(G \times H)$ -space, then $P \times_H V$ is a complex vector G -bundle over X . We obtain the following ring homomorphism being a homomorphism of $R(G)$ -modules:

$$\mu_P : R(G \times H) \rightarrow K_G(X), \quad [V] \mapsto [P \times_H V].$$

Since $\mathbf{K}_G(TX; A)$ has a $K_G(X)$ -module structure, we can formulate the following axiom.

Axiom B2. If $a \in \mathbf{K}_G(TX; A)$, $b \in K_{G \times H}(TF)$, then

$$\text{ind}_G^Y(a \cdot b) = \text{ind}_G^X(a \cdot \mu_P(\mathbf{C}\text{-ind}_{G \times H}^F(b))),$$

i. e., the diagram

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) \otimes K_{G \times H}(TF) & \xrightarrow{1 \otimes \mathbf{C}\text{-ind}_{G \times H}^F} & \mathbf{K}_G(TX; A) \otimes R(G \times H) \\ \downarrow \gamma & & \downarrow 1 \otimes \mu_P \\ \mathbf{K}_G(TY; A) & & \mathbf{K}_G(TX; A) \otimes K_G(X) \\ \downarrow \text{ind}_G^Y & \xleftarrow{\text{ind}_G^X} & \downarrow \\ K^G(A) & & \mathbf{K}_G(TX; A) \end{array}$$

is commutative

Theorem 7.7 *Let $\pi : P \rightarrow X$ be a principal right H -bundle with a left action of G commuting with H . Suppose F is a $(G \times H)$ -space. Let us denote by Y the space*

$P \times_H F$. Let $j : X_1 \rightarrow X$ and $k : F_1 \rightarrow F$ be G - and $(G \times H)$ -enclosures, respectively; let $\pi^1 : P_1 \rightarrow X_1$ be the principal H -bundle induced by j on X_1 ; assume $Y_1 := P_1 \times_H F_1$. The enclosures j and k induce G -enclosure $j * k : Y_1 \rightarrow Y$. In this situation the diagram

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) \otimes_{R(G)} K_{G \times H}(TF) & \xrightarrow{\gamma} & \mathbf{K}_G(TY; A) \\ \uparrow j_! \otimes k_! & & \uparrow (j * k)_! \\ \mathbf{K}_G(TX_1; A) \otimes_{R(G)} K_{G \times H}(TF_1) & \xrightarrow{\gamma} & \mathbf{K}_G(TY_1; A) \end{array}$$

is commutative.

Proof: Let us describe γ explicitly:

$$\begin{aligned} & \mathbf{K}_G(TX; A) \otimes K_{G \times H}(TF) \rightarrow \mathbf{K}_G(TX; A) \otimes K_{G \times H}(P \times TF) \cong \\ & \quad \uparrow j_! \otimes k_! \quad \boxed{1} \quad \uparrow \varepsilon \quad \boxed{2} \\ & \mathbf{K}_G(TX_1; A) \otimes K_{G \times H}(TF_1) \rightarrow \mathbf{K}_G(TX_1; A) \otimes K_{G \times H}(P_1 \times TF_1) \cong \\ & \quad \cong \mathbf{K}_G(TX; A) \otimes K_G(P \times_H TF) \rightarrow \\ & \quad \quad \quad \boxed{2} \quad \beta \uparrow \quad \boxed{3} \\ & \cong \mathbf{K}_G(TX_1; A) \otimes K_G(P_1 \times_H TF_1) \rightarrow \\ & \quad \rightarrow \mathbf{K}_G(\pi_1^* TX; A) \otimes K_G(P \times_H TF) \rightarrow \\ & \quad \quad \quad \boxed{3} \quad \alpha \uparrow \quad \boxed{4} \\ & \rightarrow \mathbf{K}_G((\pi^1)_1^* TX_1; A) \otimes K_G(P_1 \times_H TF_1) \rightarrow \\ & \quad \rightarrow \mathbf{K}_G((\pi_1^* TX) \times (P \times_H TF); A) \rightarrow \\ & \quad \quad \quad \boxed{4} \\ & \rightarrow \mathbf{K}_G((\pi^1)_1^* TX_1 \times (P_1 \times_H TF_1); A) \rightarrow \\ & \quad \rightarrow \mathbf{K}_G(\pi_1^* TX \oplus (P \times_H TF); A) = \mathbf{K}_G(TY; A) \\ & \quad \quad \quad \boxed{4} \quad \uparrow (j * k)_! \\ & \rightarrow \mathbf{K}_G((\pi^1)_1^* TX_1 \oplus (P_1 \times_H TF_1); A) = \mathbf{K}_G(TY_1; A). \end{aligned} \tag{12}$$

Let us remind the diagram, which was used for the definition of the Gysin homomorphism of an enclosure $j : X_1 \rightarrow X$:

$$\begin{array}{ccccccc} p_T^*(N_{X_1} \oplus N_{X_1}) & \xleftarrow{\psi} & TN_{X_1} & \xrightarrow{d\Phi_{X_1}} & TW_{X_1} & \xrightarrow{dj_2} & TX \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ TX_1 & & N_{X_1} & & W_{X_1} & & X \\ & \searrow p_T & \swarrow p_{N_{X_1}} & \searrow \Phi_{X_1} & & \searrow j_2 & \\ & & X_1 & \xrightarrow{j_1} & W_{X_1} & \xrightarrow{j_2} & X \end{array}$$

From the similar diagrams for $k_!$ and $(j * k)_!$ and the explicit form of the maps it follows that the square $\boxed{4}$ in (12) iff α has the following form:

$$\begin{aligned} \alpha(\sigma \otimes \rho) &= (\pi_1^*) \left\{ (dj_2)_* (d\Phi_{X_1}^{-1})^* \psi_{X_1}^* \right\} \circ \varphi_A^S(\sigma) \otimes \\ & \quad \otimes (\pi^* j_2 \times_H dk_2)_* \left((\pi^* \Phi_{X_1} \times_H d\Phi_{F_1})^{-1} \right)^* (1 \times_H \psi_{F_1})^* \varphi_C^T(\rho), \end{aligned}$$

where S and T are bundles of the form

$$S : \begin{array}{c} \pi_1^* \left((p_T^X)^* \{N_{X_1} \oplus N_{X_1}\} \right) \\ \downarrow (\pi^1)_1^* q_T^{X_1} \\ (\pi^1)_1^* (TX_1), \end{array} \quad T : \begin{array}{c} \pi^* N_{X_1} \times_H (p_T^{F_1})^* (N_{F_1} \oplus N_{F_1}) \\ \downarrow (\pi^1)^* (p_{N_{X_1}}) \times_H q_T^{F_1} \\ \pi^* X_1 \times_H TF_1 = P_1 \times_H TF_1. \end{array}$$

Hence the square $\boxed{3}$ in (12) is commutative iff the homomorphism β has the form

$$\begin{aligned} \beta(\tau \otimes \rho) &= j_!(\tau) \otimes \\ &\otimes (\pi^* j_2 \times_H dk_2)_* \left((\pi^* \Phi_{X_1} \times_H d\Phi_{F_1})^{-1} \right)^* (1 \times_H \psi_{F_1})^* \varphi_{\mathbf{C}}^T(\rho), \\ \tau &\in \mathbf{K}_G(TX_1; A), \quad \rho \in K_G(P_1 \times_H TF_1). \end{aligned}$$

In turn, the square $\boxed{2}$ in (12) is commutative iff the homomorphism ε has the form

$$\begin{aligned} \varepsilon(\tau \otimes \delta) &= j_!(\tau) \otimes \\ &\otimes (\pi^* j_2 \times_H dk_2)_* \left((\pi^* \Phi_{X_1} \times_H d\Phi_{F_1})^{-1} \right)^* (1 \times_H \psi_{F_1})^* \varphi_{\mathbf{C}}^{\tilde{T}}(\delta), \\ \tau &\in \mathbf{K}_G(TX_1; A), \quad \delta \in K_{G \times H}(P_1 \times_H TF_1), \end{aligned}$$

where \tilde{T} is the following bundle:

$$\tilde{T} : \begin{array}{c} \pi^* N_{X_1} \times (p_T^{F_1})^* (N_{F_1} \oplus N_{F_1}) \\ \downarrow (\pi^1)^* (p_N) \times q_T^{F_1} \\ P_1 \times TF_1. \end{array}$$

Suppose $\delta = [\underline{\mathbf{C}}] \hat{\otimes} \omega$, where $[\underline{\mathbf{C}}] \in K_{G \times H}(P_1)$, $\underline{\mathbf{C}}$ is the one-dimensional trivial bundle and $\omega \in K_{G \times H}(TF_1)$. Then

$$\begin{aligned} \varepsilon(\tau \otimes \delta) &= j_!(\tau) \otimes \left\{ \pi^*(j_2)_* (\Phi_{X_1}^{-1})^* [\underline{\mathbf{C}}] \hat{\otimes} k_!(\omega) \right\} = \\ &= j_!(\tau) \otimes \left\{ [\underline{\mathbf{C}}] \hat{\otimes} k_!(\omega) \right\}. \end{aligned}$$

Since the map $K_{G \times H}(TF) \rightarrow K_{G \times H}(P \times TF)$ (as well as the lower line in (12)) has the form $\omega \mapsto [\underline{\mathbf{C}}] \hat{\otimes} \omega$, we have proved the commutativity of $\boxed{1}$ in (12). \square

Let in the situation of axiom B2

$$\mathbf{C}\text{-ind}_{G \times H}^F(G) \in R(G) \subset R(G \times H).$$

Since μ_P and ind_G^X are $R(G)$ -homomorphisms, the following property holds.

Axiom B2'. (corollary of B2) If $\mathbf{C}\text{-ind}_{G \times H}^F(G) \in R(G) \subset R(G \times H)$, then

$$\text{ind}_G^Y(a \cdot b) = \text{ind}_G^X(a) \cdot \mathbf{C}\text{-ind}_G^F(b).$$

Assume in B2 $X = P$, $H = 1$. We obtain the following axiom.

Axiom B2''. (corollary of B2) If X and F are G -manifolds, then

$$\text{ind}_G^{X \times F}(a \cdot b) = \text{ind}_G^X(a) \cdot \mathbf{C}\text{-ind}_G^F(b).$$

Theorem 7.8 Suppose an index function ind_G^X satisfies A1, B1, B2, then

$$\text{ind}_G^X = \text{t-ind}_G^X.$$

Proof: Suppose in the axiom B2' F is equal to an open $(G \times H)$ -subset of the compact manifold \tilde{F} . Let $j : F \hookrightarrow \tilde{F}$. Then

$$\begin{aligned} \text{ind}_G^Y(a \cdot b) &= \text{ind}_{\tilde{G}}^{\tilde{Y}}(dJ_*)(a \cdot b) = \text{ind}_{\tilde{G}}^{\tilde{Y}}(a \cdot ((dj)_*b)) = \\ &= \text{ind}_G^X(a) \cdot \mathbf{C}\text{-ind}_{G \times H}^{\tilde{F}}((dj)_*b) = \\ &= \text{ind}_G^X(a) \cdot \mathbf{C}\text{-ind}_{G \times H}^F(b), \end{aligned} \tag{13}$$

where J is the enclosure

$$Y = P \times_H F \xrightarrow{\text{Id} \times_H j} P \times_H \tilde{F} = \tilde{Y}.$$

Indeed, let us consider the diagram

$$\begin{aligned} \mathbf{K}_G(TX; A) \otimes K_{G \times H}(TF) &\rightarrow \mathbf{K}_G(TX; A) \otimes K_{G \times H}(P \times TF) \cong \\ &\downarrow 1 \otimes (dj)_* \qquad \qquad \downarrow 1 \otimes (\text{Id} \times dj)_* \\ \mathbf{K}_G(TX; A) \otimes K_{G \times H}(T\tilde{F}) &\rightarrow \mathbf{K}_G(TX; A) \otimes K_{G \times H}(P \times T\tilde{F}) \cong \\ &\cong \mathbf{K}_G(TX; A) \otimes K_G(P \times_H TF) = \mathbf{K}_G(TX; A) \otimes K_G(T_F Y) \rightarrow \\ &\downarrow 1 \otimes (\text{Id} \times_H dj)_* \qquad \qquad \downarrow 1 \otimes \alpha_* \\ &\cong \mathbf{K}_G(TX; A) \otimes K_G(P \times_H T\tilde{F}) = \mathbf{K}_G(TX; A) \otimes K_G(T_{\tilde{F}} Y) \rightarrow \\ &\rightarrow \mathbf{K}_G(\pi_1^* TX; A) \otimes K_G \otimes K_G(T_F Y) \rightarrow \mathbf{K}_G(TY; A) \\ &\downarrow 1 \otimes \alpha_* \qquad \qquad \downarrow (dJ)_* \\ &\rightarrow \mathbf{K}_G(\pi_1^* TX; A) \otimes K_G(T_{\tilde{F}} Y) \rightarrow \mathbf{K}_G(T\tilde{Y}; A). \end{aligned}$$

This diagram is commutative. In fact, we have

$$\begin{aligned} TY &= T_F Y \oplus \pi_1^*(TX) \\ \downarrow dJ &\qquad \downarrow \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \\ T\tilde{Y} &= T_{\tilde{F}} Y \oplus \pi_1^*(TX) \end{aligned}$$

and $\alpha = \text{Id} \times_H dj$ under the identification $T_F Y = P \times_H TF$. We have proved the second equality in (13), the remaining are obvious.

Let us now take, in particular,

$$F = \mathbf{R}^n, \quad \tilde{F} = (\mathbf{R}^n)^+ = S^n, \quad H = O(n), \quad b = j!(1), \quad 1 = [\mathbf{C}],$$

where $j : \vec{0} \hookrightarrow \mathbf{R}^n$ is the natural enclosure. Then P is a principal $O(n)$ -bundle over X , the group G acts on P commuting with $O(n)$. Suppose G acts on \mathbf{R}^n in a trivial way. We form the associated real G -bundle

$$P \times_{O(n)} \mathbf{R}^n = Y \rightarrow X.$$

Let us denote by

$$i : X \rightarrow Y, \quad i = 1_X * j,$$

the enclosure of X as of the zero section. Assume in Theorem 7.7 $F = \mathbf{R}^n$, $X_1 = X$, $F_1 = pt$. Then we obtain the commutative diagram

$$\begin{array}{ccc} \mathbf{K}_G(TX; A) \otimes_{R(G)} K_{G \times O(n)}(T\mathbf{R}^n) & \xrightarrow{\gamma} & \mathbf{K}_G(TY; A) \\ \uparrow (1_X)_! \otimes j_! & & \uparrow (1_X * j)_! = i_! \\ \mathbf{K}_G(TX; A) \otimes_{R(G)} K_{G \times O(n)}(pt) & \xrightarrow{\gamma} & \mathbf{K}_G(TX; A). \end{array}$$

Since $\gamma(a \otimes 1) = a$,

$$\begin{aligned} i_!(a) &= \gamma(((1_X)_! \otimes j_!)(a \otimes 1)) = \gamma(a \otimes j_!(1)) = \\ &= a \cdot j_!(1) = a \cdot b. \end{aligned}$$

By the property of \mathbf{C} -ind indicated above,

$$\mathbf{C}\text{-ind}_{O(n)}^{\mathbf{R}^n} j_!(1) = 1, \quad \mathbf{C}\text{-ind}_{G \times O(n)}^{\mathbf{R}^n} j_!(1) = 1,$$

where G acts on \mathbf{R}^n in a trivial way. Now by the axiom B2,

$$\begin{aligned} \text{ind}_G^X &= \text{ind}_G^X(a \cdot 1) = \text{ind}_G^X(a \cdot \mu_P(1)) = \\ &= \text{ind}_G^X(a \cdot \mu_P(\mathbf{C}\text{-ind}_{G \times O(n)}^{\mathbf{R}^n}(b))) = \\ &= \text{ind}_G^Y(a \cdot b) = \text{ind}_G^Y i_!(a). \end{aligned} \tag{14}$$

Let $k : X \rightarrow Z$ be an enclosure of X in a compact G -manifold Z with the normal bundle N and a tubular neighborhood $\Phi : N \rightarrow W$. By the definition of the Gysin homomorphism, $k_! = (di_2 \circ d\Phi)_* j_!$, where $di_2 : TW \rightarrow TZ$ is an enclosure of tangent bundle and $j : X \rightarrow N$ is the enclosure of X as of the zero section in the normal bundle. In the diagram

$$\begin{array}{ccccc} \mathbf{K}_G(TX; A) & \xrightarrow{j_!} & \mathbf{K}_G(TN; A) & \xrightarrow{(di_2 \circ d\Phi)_*} & \mathbf{K}_G(TZ; A) \\ & \searrow \text{ind}_G^X & \downarrow \text{ind}_G^N & \swarrow \text{ind}_G^Z & \\ & & K^G(A) & & \end{array}$$

the left triangle is commutative by (14). The map $i_2 \cdot \Phi$ is an open enclosure. Hence by B2, the right triangle is commutative too. Therefore, $\text{ind}_G^X = \text{ind}_G^Z \circ k_!$. Hence A2 is satisfied. To complete the proof it remains to apply Theorem 7.6. \square

8 Proof of the index theorem

First of all obviously, the analytical index is an index function.

Lemma 8.1 *The analytical index a-ind satisfies the axiom A1.*

Proof: Elliptic operator over a point is a *GGA*-mapping $P : V \rightarrow W$ of projective *GGA*-modules and $[\sigma(P)] = [V] - [W] = \text{index } P \in K^G(A)$. \square

Theorem 8.2 *The index a-ind satisfies the axiom B1.*

Proof: Suppose $a \in \mathbf{K}_G(TU; A)$; $j_1 : U \rightarrow X_1$ and $j_2 : U \rightarrow X_2$ are G -enclosures; $\pi : TU \rightarrow U$ is the natural projection. Let the sequence

$$0 \rightarrow \pi^* E \xrightarrow{\rho} \pi^* F \rightarrow 0$$

be exact for $x \in U \setminus L$, $|\xi| > c$ (point and (co)vector). Suppose

$$\begin{aligned} \alpha : E|_{U \setminus L} &\cong (U \setminus L) \times N, & \beta : F|_{U \setminus L} &\cong (U \setminus L) \times N, \\ \rho &= (\pi^* \beta)^{-1} (\pi^* \alpha), \end{aligned}$$

L is some G -invariant compact set. Then it is possible to assume the symbols $\sigma_1 \in \text{Smb}_0(X_1, E_1, F_1)$, $\sigma_2 \in \text{Smb}_0(X_2, E_2, F_2)$ be as follows. Suppose

$$E_1 = E \cup_{j_1 \alpha} (X_1 \setminus j_1 L) \times N, \quad E_2 = E \cup_{j_2 \alpha} (X_1 \setminus j_2 L) \times N.$$

Let the similar equalities hold for F_1 and F_2 , and $\sigma_1 = \rho \cup_{j_1} \text{Id}$, $\sigma_2 = \rho \cup_{j_2} \text{Id}$. Let us pass to construction of operators \widetilde{D}_1 and \widetilde{D}_2 , which represent these symbols in $\text{Int}_0(X_1; E_1, F_1)$ $\text{Int}_0(X_2; E_2, F_2)$, respectively. Let us take a trivializing cover, a partition of unity and smoothing functions on U . Pull them back on $j_1 U$ and $j_2 U$, and then complete these collections of open sets (to obtain covers) by some open sets not intersecting with $j_1 L$ and $j_2 L$, respectively. By our symbols and with the help of this data let us construct in the usual way (non-invariant) operators $D_1, D_2 \in \text{CZ}_0$, and then

$$\widetilde{D}_1 = \text{Av}_G D_1 \in \text{Int}_0(X_1), \quad \widetilde{D}_2 = \text{Av}_G D_2 \in \text{Int}_0(X_2).$$

It is necessary to check up the equality

$$\text{index } \widetilde{D}_1 = \text{index } \widetilde{D}_2 \in K^G(A).$$

Since L is invariant, the averaging over this set is the same for both operators. Since the operators have the order 0, we compute index in L_2 -spaces. For these spaces

$$L_2(X_1, E_1) \cong L_2(j_1 L, E_1|_{j_1 L}) \oplus L_2(X_1 \setminus j_1 L, E_1|_{X_1 \setminus j_1 L})$$

and

$$\widetilde{D}_1 : L_2(X_1 \setminus j_1 L, E_1|_{X_1 \setminus j_1 L}) \cong L_2(X_1 \setminus j_1 L, E_1|_{X_1 \setminus j_1 L})$$

(this is the identity operator). Similar relations hold for \widetilde{D}_2 . On the second summand of the decomposition of L_2 we have the commutative diagram

$$\begin{array}{ccc} \Gamma(E_1|_{j_1 L}) & \xrightarrow{\widetilde{D}_1} & \Gamma(F_1|_{j_1 L}) \\ \downarrow (j_2 j_1^{-1}) \cong & & \cong \downarrow (j_2 j_1^{-1}) \\ \Gamma(E_2|_{j_2 L}) & \xrightarrow{\widetilde{D}_2} & \Gamma(F_2|_{j_2 L}). \end{array}$$

This diagram demonstrates the coincidence of indices. \square

Theorem 8.3 *The analytical index $a\text{-ind}$ satisfies the axiom B2.*

Proof: Consider the manifold $Y = P \times_H F$ over a compact manifold X , where $P \rightarrow X$ is the principal bundle with compact Lie group H , and F is a compact left H -manifold. The compact Lie group G acts on $X = P/H$ and $Y = P \times_H F$. Hereinafter the metrics on P , X , F , Y and on A -vector bundles are supposed to be invariant. Let us recall that we considered in the axiom B2 two elements and their product of special form:

$$a \in \mathbf{K}_G(TX; A), \quad b \in K_{G \times H}(TF), \quad a \cdot b \in \mathbf{K}_G(TY; A).$$

Let us represent these elements by symbols of elliptic operators. Let a be represented by a smooth G -symbol of order 1. Let $\mathcal{A}_1 \in \text{CZ}_1$ (not necessary invariant) have the symbol α . Let us choose a trivializing cover $\{U_j\}$ of the manifold X for P and Y with a subordinate partition of unity $\{\varphi_j^2\}$. Let us consider the operator $\mathcal{A}_1^j = \varphi_j \mathcal{A}_1 \varphi_j$ on U_j . Suppose $p : Y \rightarrow X$ is the projection, and $Y_j = p^{-1}(U_j) \cong U_j \times F$. The lifting $\tilde{\mathcal{A}}_1^j$ of \mathcal{A}_1^j to Y_j is a tensor product. Hence it belongs to $\text{Int}_1(Y_j)$. Extend this lifting by zero and get the element of $\text{Int}_1(Y)$. Let us average over G :

$$\tilde{\mathcal{A}} = \text{Av}_G(\sum \tilde{\mathcal{A}}_1^j) \in \text{Int}_1(Y).$$

Then $\sigma_1(\tilde{\mathcal{A}}) = \tilde{\alpha}$, where the lifting $\tilde{\alpha}$ of the symbol α is defined globally. This follows from the invariance of the metric: for the decomposition of cotangent space Y into the vertical component η horizontal component ξ we assume $\tilde{\alpha}(\xi, \eta) = \alpha(\xi)$.

Let us restrict $\tilde{\mathcal{A}}_1^j$ on sections, which are constants along the fibers of Y , then we receive \mathcal{A}_1^j . Therefore the restriction \mathcal{A} on the space of these sections is a G -invariant operator $\mathcal{A} = \text{Av}_G(\sum \mathcal{A}_1^j) \in \text{CZ}_1(X)$ with symbol α .

Let $b = [\beta] \in K_{G \times H}(TF)$, $\beta = \sigma(\mathcal{B})$, $\mathcal{B} \in \text{CZ}_1(F)$ is a $(G \times H)$ -operator. Let $\tilde{\mathcal{B}}_1$ be the operator over $P \times F$, obtained by a lifting of \mathcal{B} . Since this operator is $(G \times H)$ -invariant, it induces a G -invariant operator $\tilde{\mathcal{B}}$ over $Y = P \times_H F$ by the restriction of $\tilde{\mathcal{B}}_1$ on the constant sections along fibers of $P \times F \rightarrow Y$. Since P is locally trivial, the restriction $\tilde{\mathcal{B}}_j = \tilde{\mathcal{B}}|_{U_j}$ over $Y_j = p^{-1}(U_j) \cong U_j \times F$ is the lifting of \mathcal{B} . Then $\tilde{\mathcal{B}}_j \in \text{Int}_1(Y_j)$, $\tilde{\mathcal{B}} \in \text{Int}_1(Y)$. Suppose $\tilde{\beta} = \sigma_1(\tilde{\mathcal{B}})$, $\tilde{\beta}(\xi, \eta) = \beta(\eta)$, where ξ is the horizontal component and η is the vertical one. There is a G -invariant operator

$$D = \begin{pmatrix} \tilde{\mathcal{A}} & -\tilde{\mathcal{B}}^* \\ \tilde{\mathcal{B}} & \tilde{\mathcal{A}}^* \end{pmatrix} \in \text{Int}_1(Y, E^0 \hat{\oplus} G^0 \oplus E^1 \hat{\oplus} G^1, E^0 \hat{\oplus} G^1 \oplus E^1 \hat{\oplus} G^0),$$

$$\sigma(D) = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta}^* \\ \tilde{\beta} & \tilde{\alpha}^* \end{pmatrix},$$

and $[\sigma(D)] = [\alpha][\beta] = a \cdot b$ by the definition of the multiplication γ . Let us calculate now index D . Since $\tilde{\mathcal{B}}$ is “of complex origins”, its kernel and cokernel are invariant modules from $\mathcal{P}(A)$ and $\text{index } \tilde{\mathcal{B}} = [\text{Ker } \tilde{\mathcal{B}}] - [\text{Coker } \tilde{\mathcal{B}}] \in K^G(A)$. The operator $\tilde{\mathcal{B}}$ is the extension of \mathcal{B} to fibers; $\text{Ker } \tilde{\mathcal{B}}$ consists of those smooth sections, which lay in $\text{Ker } \mathcal{B}_x$ for each fiber Y_x . Here \mathcal{B}_x is the operator over Y_x acting as \mathcal{B} over the standard fiber F . Hence $\text{Ker } \tilde{\mathcal{B}}$ is the space of smooth sections of the vector bundle $K_{\mathcal{B}} = P \times_H \text{Ker } \mathcal{B}$ over X . Since $\tilde{\mathcal{A}}$

and $\tilde{\mathcal{B}}$ commute, $\tilde{\mathcal{A}}$ induces an operator C^K on the sections $K_{\mathcal{B}}$. From the definition of $\tilde{\mathcal{A}}$ it follows that

$$C^K = \text{Av}_G \left(\sum \varphi_j C_j^K \varphi_j \right),$$

where C_j^K is the operator induced by $\tilde{\mathcal{A}}_1^j$ on $K_{\mathcal{B}}|_{U_j}$. By the definition of $\tilde{\mathcal{A}}_1^j$, this means that $C_j^K = \mathcal{A}_1^j \otimes \text{Id}_{K_{\mathcal{B}}}$. Therefore

$$C_j^K \in \text{Int}_1, \quad C^K \in \text{Int}_1, \quad \sigma(C^K) = \alpha \otimes \text{Id}_{K_{\mathcal{B}}}.$$

Thus C^K is an elliptic G -invariant operator on X , $[\sigma(C^K)] = a[K_{\mathcal{B}}] \in \mathbf{K}_G(TX; A)$, and

$$\text{a-ind}^X a[K_{\mathcal{B}}] = \text{index}^X C^K \in K^G(A).$$

Similarly, if $L_{\mathcal{B}} = P \times_H \text{Coker } \mathcal{B}$, then $\sigma(C^L) = \alpha \otimes \text{Id}_{L_{\mathcal{B}}}$,

$$[\sigma(C^L)] = a[L_{\mathcal{B}}] \in \mathbf{K}_G(TX; A),$$

$$\text{a-ind}^X a[L_{\mathcal{B}}] = \text{index}^X C^L \in K^G(A),$$

and

$$\begin{aligned} \text{index}^X C^K - \text{index}^X C^L &= \\ &= \text{a-ind}^X (a([K_{\mathcal{B}}] - [L_{\mathcal{B}}])) = \\ &= \text{a-ind}^X (a \mu_P(\mathbf{C}\text{-ind}_{G \times H}^F b)) \in K^G(A). \end{aligned}$$

It remains to show that

$$\text{index}^Y D = \text{index}^X C^K - \text{index}^X C^L.$$

By the definition of the index of an A -Fredholm operator,

$$\text{index}^X C^K = [N_0^K] - [N_1^K], \quad C^K : M_0^K \cong M_1^K.$$

Let us remark the following. If

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}, \quad F_1 : M_0 \cong M_1, \quad F_2 : N_0 \rightarrow N_1,$$

is a decomposition for GGA -Fredholm operator, $\text{index } F = [N_0] - [N_1]$, then $\text{index } (F^*) = -\text{index } F$ and

$$-\text{index}^X C^L = \text{index}^X (C^L)^*,$$

where $(C^L)^*$ is constructed in the same way as C^L , but instead of the operator $\tilde{\mathcal{A}}$ we take $\tilde{\mathcal{A}}^*$. Suppose,

$$\text{index}^X (C^L)^* = [N_0^{L*}] - [N_1^{L*}], \quad (C^L)^* : M_0^{L*} \cong M_1^{L*}.$$

Then by the definition of $K_{\mathcal{B}}$ and $L_{\mathcal{B}}$ as the kernels and cokernels, and C^K and $(C^L)^*$, respectively, we have

$$\begin{aligned} \tilde{\mathcal{A}} : M_0^K &\cong M_1^K, & \tilde{\mathcal{B}}(M_0^K) &= 0, \\ \tilde{\mathcal{A}}^* : M_0^{L*} &\cong M_1^{L*}, & \tilde{\mathcal{B}}^*(M_0^{L*}) &= 0. \end{aligned}$$

Further,

$$D \begin{pmatrix} M_0^K \\ \oplus \\ M_0^{L*} \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{A}}(M_0^K) \\ \oplus \\ \tilde{\mathcal{A}}^*(M_0^{L*}) \end{pmatrix} = \begin{pmatrix} M_1^K \\ \oplus \\ M_1^{L*} \end{pmatrix}, \quad (15)$$

$$D \begin{pmatrix} N_0^K \\ \oplus \\ N_0^{L*} \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{A}}(N_0^K) \\ \oplus \\ \tilde{\mathcal{A}}^*(N_0^{L*}) \end{pmatrix} \subset \begin{pmatrix} N_1^K \\ \oplus \\ N_1^{L*} \end{pmatrix}. \quad (16)$$

This is the description of the action of D on $\text{Ker } \tilde{\mathcal{B}} \oplus \text{Ker } \tilde{\mathcal{B}}^*$.

Let now $x \in \text{Ker } \tilde{\mathcal{B}}^\perp$, i. e., $(x, y) = 0$ for any y such that $\tilde{\mathcal{B}}y = 0$. For these x and y we have

$$\tilde{\mathcal{B}}(\tilde{\mathcal{A}}^*y) = \tilde{\mathcal{A}}^*(\tilde{\mathcal{B}}y) = 0, \quad (\tilde{\mathcal{A}}x, y) = (x, \tilde{\mathcal{A}}^*y) = 0.$$

Therefore $\mathcal{A}((\text{Ker } \tilde{\mathcal{B}})^\perp) \subset (\text{Ker } \tilde{\mathcal{B}})^\perp$. Similarly,

$$\mathcal{A}^*((\text{Coker } \tilde{\mathcal{B}})^\perp) \subset (\text{Coker } \tilde{\mathcal{B}})^\perp.$$

Hence,

$$D \begin{pmatrix} (\text{Ker } \tilde{\mathcal{B}})^\perp \\ \oplus \\ (\text{Coker } \tilde{\mathcal{B}})^\perp \end{pmatrix} \subset \begin{pmatrix} (\text{Ker } \tilde{\mathcal{B}})^\perp \\ \oplus \\ (\text{Coker } \tilde{\mathcal{B}})^\perp \end{pmatrix}. \quad (17)$$

From

$$D^*D = \begin{pmatrix} \tilde{\mathcal{A}}^*\tilde{\mathcal{A}} + \tilde{\mathcal{B}}^*\tilde{\mathcal{B}} & 0 \\ 0 & \tilde{\mathcal{A}}\tilde{\mathcal{A}}^* + \tilde{\mathcal{B}}\tilde{\mathcal{B}}^* \end{pmatrix} \geq \begin{pmatrix} \tilde{\mathcal{B}}^*\tilde{\mathcal{B}} & 0 \\ 0 & \tilde{\mathcal{B}}\tilde{\mathcal{B}}^* \end{pmatrix}$$

it follows that the operator D is an isomorphism on $(\text{Ker } \tilde{\mathcal{B}})^\perp \oplus (\text{Coker } \tilde{\mathcal{B}})^\perp$. Taking into account the inclusion (17), (15) and (16), we obtain the following formula for the inverse image:

$$D^{-1}((\text{Ker } \tilde{\mathcal{B}})^\perp \oplus (\text{Coker } \tilde{\mathcal{B}})^\perp) \subset (\text{Ker } \tilde{\mathcal{B}})^\perp \oplus (\text{Coker } \tilde{\mathcal{B}})^\perp. \quad (18)$$

To prove that

$$D : (\text{Ker } \tilde{\mathcal{B}})^\perp \oplus (\text{Coker } \tilde{\mathcal{B}})^\perp \rightarrow (\text{Ker } \tilde{\mathcal{B}})^\perp \oplus (\text{Coker } \tilde{\mathcal{B}})^\perp$$

is an isomorphism, it remains to check up that it is an epimorphism. By calculations as above, we obtain that $(\text{Ker } \tilde{\mathcal{B}})^\perp$ is an $\tilde{\mathcal{A}}^*$ -invariant submodule and $(\text{Coker } \tilde{\mathcal{B}})^\perp$ is an $\tilde{\mathcal{A}}$ -invariant submodule. The composition DD^* defines the isomorphism

$$DD^* : (\text{Ker } \tilde{\mathcal{B}})^\perp \oplus (\text{Coker } \tilde{\mathcal{B}})^\perp \cong (\text{Ker } \tilde{\mathcal{B}})^\perp \oplus (\text{Coker } \tilde{\mathcal{B}})^\perp.$$

By (18), D is an epimorphism.

Now we can calculate the index:

$$\begin{aligned} \text{index}^Y D &= [N_0^K \oplus N_0^{L*}] - [N_1^K \oplus N_1^{L*}] = \\ &= ([N_0^K] - [N_1^K]) + ([N_0^{L*}] - [N_1^{L*}]) = \\ &= \text{index}^X C^K + \text{index}^X C^{L*} = \\ &= \text{index}^X C^K - \text{index}^X C^L = \\ &= \text{a-ind}^X(a \cdot \mu_P(\mathbf{C}\text{-ind}_{G \times H}^F b)). \quad \square \end{aligned}$$

Theorem 8.4 *Index functions a-ind and t-ind coincide.*

Proof: From the statements proved above, it follows that we can apply Theorem 7.8.

□

9 Families and algebras

Let us remind the connection between C^* -elliptic operators and families of elliptic operators for non-equivariant case (cf. [30]). The equivariant case and the case of families of C^* -operators can be obtained just similarly.

Let M be a smooth closed manifold, X be a compact Hausdorff space, $A = C(X)$ is a unital C^* -algebra. Let us consider a family of operators elliptic along the fibers of $M \times X \rightarrow X$. More precisely, we have complex bundles $E \rightarrow M \times X$, $F \rightarrow M \times X$ and an operator D , which is continuous (e. g. after enclosure of the bundles in a trivial one) family of operators $D_x : H^s(E_x) \rightarrow H^{s-m}(F_x)$. By Theorem of Jänich [20] the index

$$\text{index}_a \{D_x\} = [\text{Ker } D_x] - [\text{Coker } D_x] \in K^0(X) = K_0(A)$$

is defined (we get bundles after a compact perturbation). On the other hand, E can be considered as an A -bundle E_A over M (similarly, F_M), and we can define an operator D_A by commutativity of the following diagram (we consider the continuous cross-sections of subbundle of product bundle of X by Sobolev space of stabilized fiber)

$$\begin{array}{ccc} \Gamma(X, \{H^s(E_x)\}) & \xrightarrow{\{D_x\}_*} & \Gamma(X, \{H^{s-m}(F_x)\}) \\ \psi \uparrow \cong & & \psi \uparrow \cong \\ H^s(E_A) & \xrightarrow{D_A} & H^{s-m}(F_A). \end{array}$$

So $\{D_x\}_*(h)(x) := D_x(h(x))$, where $h : X \rightarrow \{H^s(E_x)\}$, and locally (when $E_A = E_x \otimes A$)

$$\psi(f \otimes a)(x) = f \cdot a(x), \quad \psi(D_A(f \otimes a))(x) = D_x(f \cdot a(x)) = D_x(f) \cdot a(x).$$

Lemma 9.1 *The operator D_A is an A -elliptic operator and*

$$\text{a-ind } D_A = \text{index}_a \{D_x\}.$$

Proof: As it was demonstrated, D_A is an A -homomorphism. By calculation in local coordinates we obtain immediately, that D_A is an A - Ψ DO with the correspondent symbol. The compactness of X implies its invertibility over the complement of a compact neighborhood in the cotangent bundle, i. e. ellipticity. The coincidence of (co)kernels follows from the definition of D_A after a compact perturbation, which has no influence for indices. □

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